UNIT 3

Limits and continuity

A LIMIT is defined as a number approached by the function as an independent function's variable approaches a particular value. For instance, for a function f(x) = 4x, you can say that "The limit of f(x) as x approaches 2 is 8". To be symbolic, it is written As;

$\lim_{x\to 2}(4x)=4\times 2=8$

CONTINUITY is another widespread topic in calculus. The easy method to test for the continuity of a function is to examine whether the graph of a function can be traced by a pen without lifting the pen from the paper. When you are doing with precalculus and calculus, a conceptual definition is almost sufficient but for higher level, a technical definition is required. You can learn a better and precise way of defining continuity by using limits.

Continuity Definition

A function is said to be continuous at a particular point if the following three conditions are satisfied.

1. f(a) is defined
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

A function is said to be continuous if you can trace its graph without lifting the pen from the paper. But a function is said to be discontinuous when it has any gap in between. Let us see the types discontinuities.

Types of Discontinuity

There are basically two types of discontinuity:

- Infinite Discontinuity
- Jump Discontinuity

Infinite Discontinuity

A branch of discontinuity wherein, a vertical asymptote is present at x = a and f(a) is not defined. This is also called as Asymptotic Discontinuities. If a function has values on both sides of an asymptote, then it cannot be connected, so it is discontinuous at the asymptote.

Jump Discontinuity

A branch of discontinuity wherein $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, but both the limits are finite. This is also called simple discontinuity or continuities of first kind.

Positive Discontinuity

A branch of discontinuity wherein a function has a pre-defined two-sided limit at x=a, but either f(x) is undefined at *a* or its value is not equal to the limit at *a*.

Limit Definition

A limit of a function is a number that a function reaches as the independent variable of the function reaches a given value. The value (say a) to which the function f(x) gets close arbitrarily as the value of the independent variable x becomes close arbitrarily to a given value a symbolized as f(x) = A.

Points to remember:

- If $\lim_{x\to a^-}$ f(x) is the expected value of f at x = a given the values of 'f' near x to the left of a. This value is known as **left-hand limit** of 'f' at a.
- If $\lim_{x\to a^+}$ f(x) is the expected value of f at x = a given the values of 'f' near x to the right of a. This value is known as the **right-hand limit** of f(x) at a.

- If the right-hand and left-hand limits coincide, we say the common value as the limit of f(x) at x = a and denote it by $\lim_{x\to a}$ f(x).

One-Sided Limit

The limit that is based completely on the values of a function taken at x -value that is slightly greater or less than a particular value. A two-sided limit $\lim_{x\to a} f(x)$ takes the values of x into account that are both larger than and smaller than *a*. A one-sided limit from the left $\lim_{x\to a^-} f(x)$ or from the right $\lim_{x\to a^-} f(x)$ takes only values of x smaller or greater than *a* respectively.

Properties of Limit

- The limit of a function is represented as f(x) reaches L as x tends to limit a, such that; $\lim_{x\to a}$f(x) = L
- The limit of the sum of two functions is equal to the sum of their limits, such that: $\lim_{x\to a}$ [f(x) + g(x)] = $\lim_{x\to a}$ f(x) + $\lim_{x\to a}$ g(x)
- The limit of any constant function is a constant term, such that, $\lim_{x\to a}$ C = C
- The limit of product of the constant and function is equal to the product of constant and the limit of the function, such that: $\lim_{x\to a}$ m f(x) = m $\lim_{x\to a}$ f(x)
- Quotient Rule: $\lim_{x\to a}$[f(x)/g(x)] = $\lim_{x\to a}$f(x)/$\lim_{x\to a}$g(x); if $\lim_{x\to a}$g(x) ≠ 0

Derivative
The rate of change of a quantity y with respect to another quantity x is called the derivative or differential coefficient of y with respect to x .

Differentiation of a Function
Let f(x) is a function differentiable in an interval [a, b]. That is, at every point of the interval, the derivative of the function exists finitely and is unique. Hence, we may define a new function g: [a, b] → R, such that, ∀ x ∈ [a, b], g(x) = f'(x).

This new function is said to be differentiation (differential coefficient) of the function f(x) with respect to x and it is denoted by df(x) / d(x) or Df(x) or f'(x).

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Differentiation 'from First Principle
Let f(x) is a function finitely differentiable at every point on the real number line. Then, its derivative is given by

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Standard Differentiations

1. d / d(x) (x^n) = nx^{n-1}, x ∈ R, n ∈ R
2. d / d(x) (k) = 0, where k is constant.

3. d / d(x) (e^x) = e^x
4. d / d(x) (a^x) = $a^x \log_e a > 0, a \neq 1$

5. $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}, x > 0$

6. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x}(\log_a e) = \dfrac{1}{x \log_e a}$

7. $\dfrac{d}{dx}(\sin x) = \cos x$

8. $\dfrac{d}{dx}(\cos x) = -\sin x$

9. $\dfrac{d}{dx}(\tan x) = \sec^2 x, x \neq (2n+1)\dfrac{\pi}{2}, n \in I$

10. $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x, x \neq n\pi, n \in I$

11. $\dfrac{d}{dx}(\sec x) = \sec x \tan x, x \neq (2n+1)\dfrac{\pi}{2}, n \in I$

12. $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x, x \neq n\pi, n \in I$

13. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}, -1 < x < 1$

14. $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}, -1 < x < 1$

15. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

16. $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$

17. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

18. $\dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = -\dfrac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

19. $\dfrac{d}{dx}(\sinh x) = \cosh x$

20. $\dfrac{d}{dx}(\cosh x) = \sinh x$

21. $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

22. $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

23. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sec h} x \tanh x$

24. $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \operatorname{cot h} x$

25. $\frac{d}{dx}(\sinh^{-1} x) = 1 / \sqrt{(1 + x^2)}$

26. $\frac{d}{dx}(\cosh^{-1} x) = 1 / \sqrt{(x^2 - 1)}$

27. $\frac{d}{dx}(\tanh^{-1} x) = 1 / (x^2 - 1)$

28. $\frac{d}{dx}(\coth^{-1} x) = 1 / (1 - x^2)$

29. $\frac{d}{dx}(\operatorname{sech}^{-1} x) = -1 / x\sqrt{(1 - x^2)}$

30. $\frac{d}{dx}(\operatorname{cosech}^{-1} x) = -1 / x\sqrt{(1 + x^2)}$

Fundamental Rules for Differentiation

(i) $\frac{d}{dx}\{cf(x)\} = c\frac{d}{dx} f(x)$, where c is a constant.

(ii) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$ (sum and difference rule)

(iii) $\frac{d}{dx}\{f(x)\, g(x)\} = f(x)\frac{d}{dx} g(x) + g(x)\frac{d}{dx} f(x)$ (product rule)

Generalization If $u_1, u_2, u_3, \ldots, u_n$ be a function of x, then

$$\frac{d}{dx}(u_1\, u_2\, u_3 \ldots u_n) = \left(\frac{du_1}{dx}\right)[u_2 u_3 \ldots u_n] + u_1\left(\frac{du_2}{dx}\right)[u_3 \ldots u_n] + u_1 u_2\left(\frac{du_3}{dx}\right)[u_4 u_5 \ldots u_n] + \ldots + [u_1 u_2 \ldots u_{n-1}]\left(\frac{du_n}{dx}\right)$$

(iv) $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx} f(x) - f(x)\frac{d}{dx} g(x)}{\{g(x)^2\}}$ (quotient rule)

(v) if d / d(x) f(x) = φ(x), then d / d(x) f(ax + b) = a φ(ax + b)

(vi) Differentiation of a constant function is zero i.e., d / d(x) (c) = 0.

Different Types of Differentiable Function

1. **Differentiation of Composite Function** (Chain Rule)
If f and g are differentiable functions in their domain, then fog is also differentiable and

(fog)' (x) = f' {g(x)} g' (x)

More easily, if y = f(u) and u = g(x), then dy / dx = dy / du * du / dx.

If y is a function of u, u is a function of v and v is a function of x. Then,

dy / dx = dy / du * du / dv * dv / dx.

2. **Differentiation Using Substitution**
In order to find differential coefficients of complicated expression involving inverse trigonometric functions some substitutions are very helpful, which are listed below .

S. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $a \cos\theta$
(ii)	$\sqrt{a^2 + x^2}$	$x = a \tan\theta$ or $a \cot\theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a \sec\theta$ or $a \operatorname{cosec}\theta$
(iv)	$\sqrt{a + x}$ and $\sqrt{a - x}$	$x = a \cos 2\theta$
(v)	$a \sin x + b \cos x$	$a = r \cos\alpha, b = r \sin\alpha$
(vi)	$\sqrt{x - \alpha}$ and $\sqrt{\beta - x}$	$x = \alpha \sin^2\theta + \beta \cos^2\theta$
(vii)	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$

3. **Differentiation of Implicit Functions**
If f(x, y) = 0, differentiate with respect to x and collect the terms containing dy / dx at one side and find dy / dx.

Shortcut for Implicit Functions For Implicit function, put d /dx {f(x, y)} = – ∂f / ∂x / ∂f / ∂y, where ∂f / ∂x is a partial differential of given function with respect to x and ∂f / ∂y means Partial differential of given function with respect to y.

4. **Differentiation of Parametric Functions**
If x = f(t), y = g(t), where t is parameter, then

dy / dx = (dy / dt) / (dx / dt) = d / dt g(t) / d / dt f(t) = g' (t) / f' (t)

5. **Differential Coefficient Using Inverse Trigonometrical Substitutions**

(i) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$ or $\cos^{-1}(1-2x^2)$

(iii) $2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) \\ \tan^{-1}\left(\dfrac{2x-x^2}{1}\right) \\ \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) \end{cases}$

(iv) $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

(v) $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$

(vi) $3\tan^{-1}x = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

(vii) $\cos^{-1}x + \sin^{-1}x = \pi/2$

(viii) $\tan^{-1}x + \cot^{-1}x = \pi/2$

(ix) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$

(x) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(xi) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{(1-x^2)(1-y^2)}\right]$

(xii) $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left[\dfrac{x \pm y}{1 \mp xy}\right]$

Logarithmic Differentiation Function
(i) If a function is the product and quotient of functions such as $y = f_1(x)\, f_2(x)\, f_3(x)\ldots / g_1(x)\, g_2(x)\, g_3(x)\ldots$, we first take algorithm and then differentiate.
(ii) If a function is in the form of exponent of a function over another function such as $[f(x)]^{g(x)}$, we first take logarithm and then differentiate.

Differentiation of a Function with Respect to Another Function
Let y = f(x) and z = g(x), then the differentiation of y with respect to z is

dy / dz = dy / dx / dz / dx = f' (x) / g' (x)

Successive Differentiations
If the function y = f(x) be differentiated with respect to x, then the result dy / dx or f' (x), so obtained is a function of x (may be a constant).

Hence, dy / dx can again be differentiated with respect of x.

The differential coefficient of dy / dx with respect to x is written as d /dx (dy / dx) = d^2y / dx^2 or f' (x). Again, the differential coefficient of d^2y / dx^2 with respect to x is written as
d / dx (d^2y / dx^2) = d^3y / dx^3 or f'"(x)……
Here, dy / dx, d^2y / dx^2, d^3y / dx^3,… are respectively known as first, second, third, … order differential coefficients of y with respect to x. These alternatively denoted by f' (x), f" (x), f"' (x), … or y_1, y_2, y_3…., respectively.

Note dy / dx = (dy / dθ) / (dx / dθ) but $d^2y / dx^2 \neq (d^2y / d\theta^2) / (d^2x / d\theta^2)$

nth Derivative of Some Functions

(i) $\frac{d^n}{dx^n}[\sin(ax+b)] = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

(ii) $\frac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

(iii) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$

(iv) $\frac{d^n}{dx^n}[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

(v) $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

(vi) $\frac{d^n}{dx^n}(a^x) = a^x (\log a)^n$

(vii) (a) $\frac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx+c+n\phi)$

(b) $\frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = r^n e^{ax}\cos(bx+c+n\phi)$

where, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

Derivatives of Special Types of Functions

(i) If $y = f(x)^{\{f(x)\}^{\dots\infty}}$, then $\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)\{1 - y\log f(x)\}}$

(ii) If $e^{g(y)} - e^{-g(y)} = 2f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{g'(y)} \cdot \frac{1}{\sqrt{1+\{f(x)\}^2}}$

(iii) If $y = \sqrt{\frac{1+g(x)}{1-g(x)}}$, then $\frac{dy}{dx} = \frac{g'(x)}{[1-g(x)]^2} \cdot \sqrt{\frac{1-g(x)}{1+g(x)}}$

(iv) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \dots\infty}}}$, then $\frac{dy}{dx} = \frac{f'(x)}{2y-1}$

(v) If $\{f(x)\}^{g(y)} = e^{f(x)-g(y)}$, then $\frac{dy}{dx} = \frac{f'(x)\log f(x)}{g'(y)\{1+\log f(x)\}^2}$

(vi) If $\{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}$, then

$$\frac{dy}{dx} = \frac{g(y)}{f(x)} \cdot \frac{f'(x)}{g'(y)}\left[\frac{f(x)\log g(y) - g(y)}{g(y)\log f(x) - f(x)}\right]$$

QUESTION BANK OF LIMIT AND DIFFERENTIATION

1. Evaluate the Given limit: $\lim_{x \to 3} x + 3$

Solution:

Given

$$\lim_{x \to 3} x + 3$$

Substituting x = 3, we get

= 3 + 3

= 6

2. Evaluate the Given limit: $\lim_{x \to \pi}\left(x - \frac{22}{7}\right)$

Solution:

Given limit:

$$\lim_{x \to \pi}\left(x - \frac{22}{7}\right)$$

Substituting x = π, we get

$\lim_{x \to \pi}\left(x - \frac{22}{7}\right)$ = (π – 22 / 7)

3. Evaluate the Given limit: $\lim_{r \to 1} \pi r^2$

Solution:

Given limit**:** $\lim_{r \to 1} \pi r^2$

Substituting r = 1, we get

$\lim_{r \to 1} \pi r^2$ **=** $\pi(1)^2$

= π

4. Evaluate the Given limit: $\lim_{x \to 4} \frac{4x+3}{x-2}$

Solution:

Given limit:

$$\lim_{x \to 4} \frac{4x+3}{x-2}$$

Substituting x = 4, we get

$\lim_{x \to 4} \frac{4x+3}{x-2}$ = [4(4) + 3] / (4 – 2)

= (16 + 3) / 2

= 19 / 2

5. Evaluate the Given limit: $\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1}$

Solution:

Given limit:

$$\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1}$$

Substituting x = -1, we get

$$\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1}$$

= $[(-1)^{10} + (-1)^5 + 1] / (-1 – 1)$

= (1 – 1 + 1) / – 2

= – 1 / 2

6. Evaluate the Given limit: $\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$

Solution:

Given limit:

$$\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$$

= $[(0 + 1)^5 – 1] / 0$

=0

Since, this limit is undefined

Substitute x + 1 = y, then x = y – 1

$$\lim_{y \to 1} \frac{(y)^5 - 1}{y - 1}$$

$$= \lim_{y \to 1} \frac{(y)^5 - 1^5}{y - 1}$$

We know that,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Hence,

$$\lim_{y \to 1} \frac{(y)^5 - 1^5}{y - 1}$$

$=5(1)^{5-1}$

$= 5(1)^4$

$= 5$

7. Evaluate the Given limit: $\lim_{x\to 2} \frac{3x^2 - x - 10}{x^2 - 4}$

Solution:

By evaluating the limit at x = 2, we get

$\lim_{x\to 2} \frac{3x^2-x-10}{x^2-4}$ = $[3(2)^2 - x - 10] / 4 - 4$

= 0

Now, by factorising numerator, we get

$\lim_{x\to 2} \frac{3x^2-x-10}{x^2-4} = \lim_{x\to 2} \frac{3x^2-6x+5x-10}{x^2-2^2}$

We know that,

$a^2 - b^2 = (a - b)(a + b)$

$= \lim_{x\to 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)}$

$= \lim_{x\to 2} \frac{(3x+5)}{(x+2)}$

By substituting x = 2, we get,

= [3(2) + 5] / (2 + 2)

= 11 / 4

8. Evaluate the Given limit: $\lim_{x\to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$

Solution:

First substitute x = 3 in the given limit, we get

$$= \lim_{x \to 3} \frac{(3)^4 - 81}{2(3)^2 - 5 \times 3 - 3}$$

= (81 – 81) / (18 – 18)

= 0 / 0

Since the limit is of the form 0 / 0, we need to factorise the numerator and denominator

$$\lim_{x \to 3} \frac{(x^2 - 9)(x^2 + 9)}{2x^2 - 6x + x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)(x^2 + 9)}{(2x + 1)(x - 3)}$$

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{(x + 3)(x^2 + 9)}{(2x + 1)}$$

Now substituting x = 3, we get

$$= \frac{(3 + 3)(3^2 + 9)}{(2 \times 3 + 1)}$$

= 108 / 7

Hence,

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = 108 / 7$$

9. Evaluate the Given limit: $\lim_{x \to 0} \frac{ax + b}{cx + 1}$

Solution:

$$\lim_{x \to 0} \frac{ax + b}{cx + 1}$$

= [a (0) + b] / c (0) + 1

= b / 1

= b

10. Evaluate the Given limit: $\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$

Solution:

$$\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} = (1-1)/(1-1)$$

$= 0$

Let the value of $z^{1/6}$ be x

$(z^{1/6})^2 = x^2$

$z^{1/3} = x^2$

Now, substituting $z^{1/3} = x^2$ we get

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \frac{x^2-1^2}{x-1}$$

We know that,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$$

$$\lim_{x\to 1}\frac{x^2-1^2}{x-1} = 2\,(1)^{2-1}$$

$= 2$

11. Evaluate the Given limit: $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c\neq 0$

Solution:

Given limit:

$$\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c\neq 0$$

Substituting x = 1

$$\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}$$

= [a $(1)^2$ + b (1) + c] / [c $(1)^2$ + b (1) + a]

= (a + b + c) / (a + b + c)

Given

$$[a+b+c\neq 0]$$

= 1

12. Evaluate the Given limit: $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Solution:

By substituting x = – 2, we get

$$\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2} = 0 / 0$$

Now,

$$\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2} = \frac{\frac{2+x}{2x}}{x+2}$$

$$= 1 / 2x$$

$$= 1 / 2(-2)$$

$$= - 1 / 4$$

13. Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax}{bx}$

Solution:

Given $\lim_{x \to 0} \frac{\sin ax}{bx}$

Formula used here

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

By applying the limits in the given expression

$$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{0}{0}$$

By multiplying and dividing by 'a' in the given expression, we get

$$\lim_{x \to 0} \frac{\sin ax}{bx} \times \frac{a}{a}$$

We get,

$$\lim_{x \to 0} \frac{\sin ax}{ax} \times \frac{a}{b}$$

We know that,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$= \frac{a}{b} \lim_{ax \to 0} \frac{\sin ax}{ax} = \frac{a}{b} \times 1$$

$$= a / b$$

14. Evaluate the given limit: $\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx}, a, b \neq 0$

Solution:

$\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx} = 0 / 0$

By multiplying ax and bx in numerator and denominator, we get

$\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx} = \lim\limits_{x \to 0} \frac{\frac{\sin ax}{ax} \times ax}{\frac{\sin bx}{bx} \times bx}$

Now, we get

$\frac{a}{b} \frac{\lim\limits_{ax \to 0} \frac{\sin ax}{ax}}{\lim\limits_{bx \to 0} \frac{\sin bx}{bx}}$

We know that,

$\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

Hence, a / b × 1

= a / b

15. Evaluate the given limit:

$$\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

Solution:

$$\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

$$\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \lim_{\pi - x \to 0} \frac{\sin(\pi - x)}{(\pi - x)} \times \frac{1}{\pi}$$

$$= \frac{1}{\pi} \lim_{\pi - x \to 0} \frac{\sin(\pi - x)}{(\pi - x)}$$

We know that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\frac{1}{\pi} \lim_{\pi - x \to 0} \frac{\sin(\pi - x)}{(\pi - x)} = \frac{1}{\pi} \times 1$$

$= 1 / \pi$

16. Evaluate the given limit:

$$\lim_{x \to 0} \frac{\cos x}{\pi - x}$$

Solution:

$$\lim_{x \to 0} \frac{\cos x}{\pi - x} = \frac{\cos 0}{\pi - 0}$$

$= 1 / \pi$

17. Evaluate the given limit:

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$$

Solution:

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\frac{0}{0}$$

Hence,

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1}$$

($\cos 2x = 1 - 2\sin^2 x$)

$$\lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\frac{\sin^2 x\times x^2}{x^2}}{\frac{\sin^2\frac{x}{2}\times\frac{x^2}{4}}{(\frac{x}{2})^2}}$$

$$= 4\,\frac{\lim_{x\to 0}\frac{\sin^2 x}{x^2}}{\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}}$$

$$= 4\,\frac{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)^2}{\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{(\frac{x}{2})^2}\right)^2}$$

We know that,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$= 4 \times 1^2 / 1^2$

$= 4$

18. Evaluate the given limit:

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$$

Solution:

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{0}{0}$$

Hence,

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}$$

$$=\frac{1}{b}\lim_{x\to 0}\times\lim_{x\to 0}(a+\cos x)$$

$$=\frac{1}{b}\times\frac{1}{\lim_{x\to 0}\frac{\sin x}{x}}\times\lim_{x\to 0}(a+\cos x)$$

We know that,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\frac{1}{b}\times(a+\cos 0)$$

$= (a + 1) / b$

19. Evaluate the given limit:

$$\lim_{x\to 0}x\sec x$$

Solution:

$$\lim_{x\to 0}x\sec x=\lim_{x\to 0}\frac{x}{\cos x}$$

$$=\lim_{x\to 0}\frac{0}{\cos 0}=\frac{0}{1}$$

$= 0$

20. Evaluate the given limit:

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}a,b,a+b\neq 0$$

Solution:

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}=\frac{0}{0}$$

Hence,

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}=\lim_{x\to 0}\frac{(\sin\frac{ax}{ax})ax+bx}{ax+(\sin\frac{bx}{bx})}$$

$$=\frac{\left(\lim_{ax\to 0}\sin\frac{ax}{ax}\right)\times\lim_{x\to 0}ax+\lim_{x\to 0}bx}{\lim_{x\to 0}ax+\lim_{x\to 0}bx\times(\lim_{bx\to 0}\sin\frac{bx}{bx})}$$

We know that,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\frac{\lim_{x\to 0}ax+\lim_{x\to 0}bx}{\lim_{x\to 0}ax+\lim_{x\to 0}bx}$$

We get,

$$=\frac{\lim_{x\to 0}(ax+bx)}{\lim_{x\to 0}(ax+bx)}$$

= 1

21. Evaluate the given limit:

$$\lim_{x\to 0}(\operatorname{cosec}x-\cot x)$$

Solution:

$$\lim_{x \to 0} (\text{cosec } x - \cot x)$$

Applying the formulas for cosec x and cot x, we get

$$\textbf{cosec } x = \frac{1}{\sin x} \textit{ and } \cot x = \frac{\cos x}{\sin x}$$

$$\lim_{x \to 0} (\text{cosec } x - \cot x) = \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$\lim_{x \to 0} (cosec\, x - cot\, x) = \lim_{x \to 0} \frac{1 - \cos x}{\sin x}$$

Now, by applying the formula we get,

$$1 - \cos x = 2 \sin^2 \frac{x}{2} \textit{ and } \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$\lim_{x \to 0} (\text{cosec } x - \cot x) = \lim_{x \to 0} \frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$\lim_{x \to 0} (cosec\, x - cot\, x) = \lim_{x \to 0} \tan \frac{x}{2}$$

= 0

22. Evaluate the given limit:

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$$

Solution:

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}} = \frac{0}{0}$$

Let $x - (\pi / 2) = y$

Then, $x \to (\pi/2) = y \to 0$

Now, we get

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}} = \lim_{y\to 0} \frac{\tan 2(y+\frac{\pi}{2})}{y}$$

$$= \lim_{y\to 0} \frac{\tan(2y+\pi)}{y}$$

$$= \lim_{y\to 0} \frac{\tan(2y)}{y}$$

We know that,

$\tan x = \sin x / \cos x$

$$= \lim_{y\to 0} \frac{\sin 2y}{y \cos 2y}$$

By multiplying and dividing by 2, we get

$$= \lim_{y\to 0} \frac{\sin 2y}{2y} \times \frac{2}{\cos 2y}$$

$$= \lim_{2y\to 0} \frac{\sin 2y}{2y} \times \lim_{y\to 0} \frac{2}{\cos 2y}$$

$= 1 \times 2 / \cos 0$

$= 1 \times 2 / 1$

$= 2$

23.

Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x+3 & x \le 0 \\ 3(x+1) & x > 0 \end{cases}$

Solution:

Given function is $f(x) = \begin{cases} 2x+3 & x \le 0 \\ 3(x+1) & x > 0 \end{cases}$

$\lim\limits_{x \to 0} f(x)$:

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0}(2x+3)$

$= 2(0) + 3$

$= 0 + 3$

$= 3$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} 3(x+1)$

$=3(0+1)$

$= 3(1)$

$= 3$

Hence, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} f(x) = 3$

Now, for $\lim_{x\to 1} f(x)$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} 3(x+1)$$

$= 3\ (1 + 1)$

$= 3\ (2)$

$= 6$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 3(x+1)$$

$= 3\ (1 + 1)$

$= 3\ (2)$

$= 6$

Hence, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 6$

$\lim_{x\to 0} f(x) = 3$ and $\lim_{x\to 1} f(x) = 6$

24. Find

$\lim_{x\to 1} f(x)$ **, where**

$$f(x) = \begin{cases} x^2 - 1 & x \le 1 \\ -x^2 - 1 & x > 1 \end{cases}$$

Solution:

Given function is:

$$f(x) = \begin{cases} x^2 - 1 & x \le 1 \\ -x^2 - 1 & x > 1 \end{cases}$$

$\lim_{x\to 1} f(x)$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} x^2 - 1$$

$= 1^2 - 1$

$= 1 - 1$

$= 0$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} (-x^2 - 1)$$

$= (-1^2 - 1)$

$= -1 - 1$

$= -2$

We find,

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

Hence, $\lim_{x\to 1} f(x)$ does not exist

25. Evaluate

$\lim_{x\to 0} f(x)$ **, where f(x) =**

$$\begin{cases} \frac{|x|}{x}, x \neq 0 \\ x \\ 0, x = 0 \end{cases}$$

Solution:

Given function is $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ x & \\ 0, & x = 0 \end{cases}$

We know that,

$\lim_{x \to a} f(x)$ exists only when $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

Now, we need to prove that: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

We know,

|x| = x, if x >= -x, if x < 0

Hence,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x}$

$= \lim_{x \to 0} \frac{-x}{x} = \lim_{x \to 0}(-1)$

= -1

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x}$

$= \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0}(1)$

= 1

We find here,

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence, $\lim_{x\to 0} f(x)$ does not exist.

26. Find

$\lim_{x\to 0} f(x)$, **where f (x) =**

$$\begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution:

Given function is:

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\lim_{x\to 0} f(x)$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{|x|}$$

$$= \lim_{x\to 0} \frac{x}{-x} = \lim_{x\to 0} \frac{1}{-1}$$

$= -1$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{|x|}$$

$= \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} (1)$

$= 1$

We find here,

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence, $\lim_{x \to 0} f(x)$ does not exist.

27. Find

$\lim_{x \to 5} f(x)$, where

$f(x) = |x| - 5$

Solution:

Given function is:

$f(x) = |x| - 5$

$\lim_{x \to 5} f(x)$:

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} |x| - 5$

$= \lim_{x \to 5} (x - 5) = 5 - 5$

$= 0$

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} |x| - 5$

$= \lim_{x \to 5} (x - 5)$

$= 5 - 5$

$= 0$

Hence, $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = \lim_{x \to 5} f(x) = 0$

28. Suppose

$$f(x) = \begin{cases} a + bx, x < 1 \\ 4, \quad x = 1 \\ b - ax \ x > 1 \end{cases}$$ **and if**

$\lim_{x \to 1} f(x) = f(1)$ **what are possible values of a and b**

Solution:

Given function is:

$$f(x) = \begin{cases} a + bx, x < 1 \\ 4, x = 1 \\ b - ax, x > 1 \end{cases}$$ and

$\lim_{x \to 1} f(x) = f(1)$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} a + bx$

$= a + b\,(1)$

$= a + b$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} b - ax$

$= b - a\,(1)$

$= b - a$

Here,

$f(1) = 4$

Hence, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = f(1)$

Then, $a + b = 4$ and $b - a = 4$

By solving the above two equations, we get,

$a = 0$ and $b = 4$

Therefore, the possible values of a and b is 0 and 4 respectively

29. Let a1, a2,………an be fixed real numbers and define a function

f (x) = (x – a1) (x – a2) ……. (x – an).

What is

$\lim_{x \to a_1} f(x)$? **For some a ≠ a1, a2, ……. an, compute**

$\lim_{x \to a} f(x)$

Solution:

Given function is:

$f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$

$\lim_{x \to a_1} f(x)$:

$$\lim_{x \to a_1} f(x) = \lim_{x \to a_1} [(x - a_1)(x - a_2) \ldots (x - a_n)]$$

$$= [\lim_{x \to a_1}(x - a_1)] \left[\lim_{x \to a_1}(x - a_2)\right] \ldots \left[\lim_{x \to a_1}(x - a_n)\right]$$

We get,

$$= (a_1 - a_1)(a_1 - a_2) \ldots (a_1 - a_n) = 0$$

Hence, $\lim_{x \to a_1} f(x) = 0$

$\lim_{x \to a} f(x)$:

$$\lim_{x \to a} f(x) = \lim_{x \to a} [(x - a_1)(x - a_2) \ldots (x - a_n)]$$

$$= [\lim_{x \to a}(x - a_1)] \left[\lim_{x \to a}(x - a_2)\right] \ldots \left[\lim_{x \to a}(x - a_n)\right]$$

We get,

$$= (a - a_1)(a - a_2) \ldots\ldots (a - a_n)$$

Hence, $\lim_{x \to a} f(x) = (a - a_1)(a - a_2) \ldots (a - a_n)$

Therefore, $\lim_{x \to a_1} f(x) = 0$ and $\lim_{x \to a} f(x) = (a - a_1)(a - a_2) \ldots (a - a_n)$

30. If $f(x)=\begin{cases}|x|+1, x<0\\ 0, \quad x=0\\ |x|-1, x>0\end{cases}$ **For what value (s) of a does** $\lim_{x\to a} f(x)$ **exists?**

Solution:

Given function is:

$$f(x)=\begin{cases}|x|+1, x<0\\ 0, \quad x=0\\ |x|-1, x>0\end{cases}$$

There are three cases.

Case 1:

When $a=0$

$\lim_{x\to 0} f(x)$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(|x|+1)$$

$$= \lim_{x\to 0}(-x+1) = -0+1$$

$$= 1$$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (|x| - 1)$

$= \lim_{x \to 0} (x - 1) = 0 - 1$

= -1

Here, we find

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence, $\lim_{x \to 0} f(x)$ does not exit.

Case 2:

When a < 0

$\lim_{x \to a} f(x)$:

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (|x| + 1)$

$= \lim_{x \to a} (-x + 1) = -a + 1$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} (|x| + 1)$

$= \lim_{x \to a} (-x + 1) = -a + 1$

Hence, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = -a + 1$

Therefore, lim (f(x)) exists at x = a and a < 0

Case 3:

When a > 0

$\lim_{x \to a} f(x)$:

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (|x| - 1)$

$= \lim_{x \to a}(x - 1) = a - 1$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} (|x| - 1)$

$= \lim_{x \to a}(x - 1) = a - 1$

Hence, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = a - 1$

Therefore, lim (f (x)) exists at x = a when a > 0

31. If the function f(x) satisfies $\lim_{x \to 1} \frac{f(x) - 2}{x^2 - 1} = \pi$**, evaluate** $\lim_{x \to 1} f(x)$

Solution:

Given function that f (x) satisfies $\lim\limits_{x\to 1}\dfrac{f(x)-2}{x^2-1}=\pi$

$$\frac{\lim\limits_{x\to 1} f(x)-2}{\lim\limits_{x\to 1} x^2-1}=\pi$$

$$\lim_{x\to 1}(f(x)-2)=\pi(\lim_{x\to 1}(x^2-1))$$

Substituting $x=1$, we get,

$$\lim_{x\to 1}(f(x)-2)=\pi(1^2-1)$$

$$\lim_{x\to 1}(f(x)-2)=\pi(1-1)$$

$$\lim_{x\to 1}(f(x)-2)=0$$

$$\lim_{x\to 1} f(x)-\lim_{x\to 1} 2=0$$

$$\lim_{x\to 1} f(x)-2=0$$

$$=2$$

32. If $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m. & x>1 \end{cases}$ **For what integers m and n does both** $\lim\limits_{x\to 0} f(x)$ **and** $\lim\limits_{x\to 1} f(x)$ **exist?**

Solution:

Given function is $$f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m. & x>1 \end{cases}$$

$\lim\limits_{x\to 0} f(x)$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(mx^2+n)$$

$= m\,(0) + n$

$= 0 + n$

$= n$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(nx+m)$$

$= n\,(0) + m$

$= 0 + m$

$= m$

Hence,

$\lim\limits_{x\to 0} f(x)$ exists if n = m.

Now,

$\lim\limits_{x\to 1} f(x)$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(nx+m)$$

$= n\,(1) + m$

$= n + m$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(nx^3 + m)$$

$$= n\,(1)^3 + m$$

$$= n\,(1) + m$$

$$= n + m$$

Therefore $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x)$

Hence, for any integral value of m and n $\lim_{x\to 1} f(x)$ exists.

Exercise 13.2 page no: 312

1. Find the derivative of x2– 2 at x = 10

Solution:

Let f (x) = x2 – 2

From first principle

From first principle

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(10)}{h}$$

Put $x = 10$, we get

$$f'(10) = \lim_{h\to 0}\frac{f(10+h) - f(10)}{h}$$

$$= \lim_{h\to 0}\frac{[(10+h)^2 - 2] - (10^2 - 2)}{h}$$

$$= \lim_{h\to 0}\frac{10^2 + 2\times 10\times h + h^2 - 2 - 10^2 + 2}{h}$$

$$= \lim_{h \to 0} \frac{20h + h^2}{h}$$

$$= \lim_{h \to 0} (20 + h)$$

$$= 20 + 0$$

$$= 20$$

2. Find the derivative of x at x = 1.

Solution:

Let f (x) = x

Then,

From first principle

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(10)}{h}$$

Let f (x) = x

From first principle

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(10)}{h}$$

Put x = 1, we get

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)-1}{h}$$

$$= \lim_{h\to 0} \frac{1+h-1}{h}$$

$$= \lim_{h\to 0} \frac{h}{h}$$

$$= \lim_{h\to 0} 1$$

$$= 1$$

3. Find the derivative of 99x at x = l00.

Solution:

Let f (x) = 99x,

From first principle

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(10)}{h}$$

Put $x = 100$, we get

$$f'(100) = \lim_{h\to 0} \frac{f(100+h)-f(100)}{h}$$

$$= \lim_{h\to 0} \frac{99(100+h)-99\times 100}{h}$$

$$= \lim_{h\to 0} \frac{99\times 100+99h-99\times 100}{h}$$

$$= \lim_{h\to 0} \frac{99\times h}{h}$$

$$= \lim_{h\to 0} 99$$

= 99

4. Find the derivative of the following functions from first principle

(i) x3 – 27

(ii) (x – 1) (x – 2)

(iii) 1 / x2

(iv) x + 1 / x – 1

Solution:

(i) Let f (x) = x3 – 27

From first principle

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$

$$= \lim_{h \to 0}(h^2 + 3x^2 + 3xh)$$

$$= 0 + 3x^2$$

$$= 3x^2$$

(ii) Let f (x) = (x – 1) (x – 2)

From first principle

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2) - (x^2 - 2x - x + 2)}{h}$$

$$= \lim_{h \to 0} \frac{hx + hx + h^2 - 2h - h}{h}$$

$$= \lim_{h \to 0}(h + 2x - 3)$$

$$= 0 + 2x - 3$$

$$= 2x - 3$$

(iii) Let f (x) = 1 / x2

From first principle, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - x^2 - h^2 - 2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-h^2 - 2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0} \left[\frac{-h - 2x}{x^2(x+h)^2}\right]$$

$$= (0 - 2x) / [x^2 (x + 0)^2]$$

$$= (-2 / x^3)$$

(iv) Let f (x) = x + 1 / x – 1

From first principle, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{h(x-1)(x+h-1)}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx + x - x + h - 1)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h \to 0} \frac{-2h}{h(x-1)(x+h-1)}$$

$$= \lim_{h \to 0} \frac{-2}{(x-1)(x+h-1)}$$

$$= -\frac{2}{(x-1)(x-1)}$$

$$= -\frac{2}{(x-1)^2}$$

5. For the function $f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots\frac{x^{2}}{2}+x+1$ **.Prove that f' (1) =100 f' (0).**

Solution:

Given function is:

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots\frac{x^{2}}{2}+x+1$$

By differentiating both sides, we get

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+\cdots+\frac{x^{2}}{2}+x+1\right]$$

$$=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+\cdots+\frac{d}{dx}\left(\frac{x^{2}}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

We know that,

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

$$\therefore\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+\cdots+\frac{2x}{2}+1+0$$

$$f'(x) = x^{99} + x^{98} + \cdots + x + 1$$

At $x = 0$, we get

$f'(0) = 0 + 0 + \ldots + 0 + 1$

$f'(0) = 1$

At $x = 1$, we get

$f'(1) = 1^{99} + 1^{98} + \ldots + 1 + 1 = [1 + 1 \ldots + 1]$ 100 times $= 1 \times 100 = 100$

Hence, $f'(1) = 100 f'(0)$

6. Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$ **for some fixed real number a.**

Solution:

Given function is:

f(x) = $x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}\left(x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n\right)$$

$$= \frac{d}{dx}(x^n) + a\frac{d}{dx}(x^{n-1}) + a^2\frac{d}{dx}(x^{n-2}) + \cdots + a^{n-1}\frac{d}{dx}(x) + a^n\frac{d}{dx}(1)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1} + a^n(0)$$

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1}$$

7. For some constants a and b, find the derivative of
(i) (x − a) (x − b)

(ii) (ax2 + b)2

(iii) x – a / x – b

Solution:

(i) (x – a) (x – b)

Let f (x) = (x – a) (x – b)

$f(x) = x^2 - (a + b)x + ab$

Now, by differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(x^2 - (a+b)x + ab)$$

$$= \frac{d}{dx}(x^2) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$f'(x) = 2x - (a + b) + 0$

$= 2x - a - b$

(ii) (ax2 + b)2

Let $f(x) = (ax^2 + b)^2$

$f(x) = a^2x^4 + 2abx^2 + b^2$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2)$$

$$f'(x) = \frac{d}{dx}(x^4) + (2ab)\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$f'(x) = a^2 \times 4x^3 + 2ab \times 2x + 0$

$= 4a^2x^3 + 4abx$

= 4ax(ax2 + b)

(iii) x – a / x – b

$$\text{Let } f(x) = \frac{(x-a)}{(x-b)}$$

By differentiating both sides and using quotient rule, we get

$$f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2}$$

By further calculation, we get

$$= \frac{x-b-x+a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

8. Find the derivative of $\frac{x^n - a^n}{x - a}$ **for some constant a.**

Solution:

$$\text{Let } f(x) = \frac{x^n - a^n}{x-a}$$

By differentiating both sides and using quotient rule, we get

$$f'(x) = \frac{d}{dx}\left(\frac{x^n - a^n}{x-a}\right)$$

$$f'(x) = \frac{(x-a)\frac{d}{dx}(x^n - a^n) - (x^n - a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

By further calculation, we get

$$= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

9. Find the derivative of

(i) 2x – 3 / 4

(ii) (5x3 + 3x – 1) (x – 1)

(iii) x-3 (5 + 3x)

(iv) x5 (3 – 6x-9)

(v) x-4 (3 – 4x-5)

(vi) (2 / x + 1) – x2 / 3x – 1

Solution:

(i)

Let f (x) = 2x – 3 / 4

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}\left(2x - \frac{3}{4}\right)$$

$$= 2\frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{3}{4}\right)$$

$= 2 - 0$

$= 2$

(ii)

Let f (x) = $(5x^3 + 3x – 1)(x – 1)$

By differentiating both sides and using the product rule, we get

$$f'(x) = (5x^3 + 3x - 1)\frac{d}{dx}(x - 1) + (x - 1)\frac{d}{dx}(5x^3 + 3x + 1)$$

$$= (5x^3 + 3x - 1) \times 1 + (x - 1) \times (15x^2 + 3)$$

$= (5x^3 + 3x - 1) + (x - 1)(15x^2 + 3)$

$= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$

$= 20x^3 - 15x^2 + 6x - 4$

(iii)

Let f (x) = x^{-3} (5 + 3x)

By differentiating both sides and using Leibnitz product rule, we get

$$f'(x) = x^{-3}\frac{d}{dx}(5+3x)+(5+3x)\frac{d}{dx}\left(x^{-3}\right)$$

$$= x^{-3}(0+3)+(5+3x)\left(-3x^{-3-1}\right)$$

By further calculation, we get

$$= x^{-3}(3)+(5+3x)\left(-3x^{-4}\right)$$

$$= 3x^{-3}-15x^{-4}-9x^{-3}$$

$$= -6x^{-3}-15x^{-4}$$

$$= -3x^{-3}\left(2+\frac{5}{x}\right)$$

$$= \frac{-3x^{-3}}{x}(2x+5)$$

$$= \frac{-3}{x^{4}}(5+2x)$$

(iv)

Let $f(x) = x^5 (3 - 6x^{-9})$

By differentiating both sides and using Leibnitz product rule, we get

$$f'(x) = x^5 \frac{d}{dx}(3-6x^{-9}) + (3-6x^{-9})\frac{d}{dx}(x^5)$$

$$= x^5\{0-6(-9)x^{-9-1}\} + (3-6x^{-9})(5x^4)$$

By further calculation, we get

$$= x^5(54x^{-10}) + 15x^4 - 30x^{-5}$$

$$= 54x^{-5} + 15x^4 - 30x^{-5}$$

$$= 24x^{-5} + 15x^4$$

$$= 15x^4 + \frac{24}{x^5}$$

(v)

Let $f(x) = x^{-4} (3 - 4x^{-5})$

By differentiating both sides and using Leibnitz product rule, we get

$$f'(x) = x^{-4} \frac{d}{dx}(3-4x^{-5}) + (3-4x^{-5})\frac{d}{dx}(x^{-4})$$

$$= x^{-4}\{0-4(-5)x^{-5-1}\} + (3-4x^{-5})(-4)x^{-4-1}$$

By further calculation, we get

$$= x^{-4}(20x^{-6}) + (3-4x^{-5})(-4x^{-5})$$

$$= 20x^{-10} - 12x^{-5} + 16x^{-10}$$

$$= 36x^{-10} - 12x^{-5}$$

$$= -\frac{12}{x^5} + \frac{36}{x^{10}}$$

(vi)

Let $f(x) = \dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

By differentiating both sides we get,

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1} - \frac{x^2}{3x-1}\right)$$

Using quotient rule we get,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x) - (x^2)\times 3}{(3x-1)^2}\right]$$

$$= -\frac{2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right]$$

$$= -\frac{2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

10. Find the derivative of cos x from first principle

Solution:

Let f (x) = cos x

Accordingly, f (x + h) = cos (x + h)

By first principle, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

So, we get

$$= \lim_{h\to 0}\frac{1}{h}[\cos(x+h)-\cos(x)]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]$$

By further calculation, we get

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}-\sin\left(\frac{2x+h}{2}\right)\times\lim_{h\to 0}\frac{\sin(\frac{h}{2})}{\frac{h}{2}}$$

$$= -\sin\left(\frac{2x+0}{2}\right)\times 1$$

= - sin (2x / 2)

= - sin (x)

11. Find the derivative of the following functions:

(i) sin x cos x

(ii) sec x

(iii) 5 sec x + 4 cos x

(iv) cosec x

(v) 3 cot x + 5 cosec x

(vi) 5 sin x – 6 cos x + 7

(vii) 2 tan x – 7 sec x

Solution:

(i) sin x cos x

Let f (x) = sin x cos x

Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h}$$

$$= \lim_{h\to 0} \frac{1}{2h}\left[2\sin(x+h)\cos(x+h) - 2\sin x \cos x\right]$$

$$= \lim_{h\to 0} \frac{1}{2h}\left[\sin 2(x+h) - \sin 2x\right]$$

$$= \lim_{h\to 0} \frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2} \cdot \sin\frac{2x+2h-2x}{2}\right]$$

By further calculation, we get

$$= \lim_{h\to 0} \frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\cos(2x+h)\sin h\right]$$

$$= \lim_{h\to 0}\cos(2x+h) . \lim_{h\to 0}\frac{\sin h}{h}$$

$$= \cos(2x+0) . 1$$

$$= \cos 2x$$

(ii) sec x

Let f (x) = sec x

= 1 / cos x

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

Using quotient rule, we get

$$f'(x) = \frac{\cos x \frac{d}{dx}(1) - 1\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \times 0 - (-\sin x)}{\cos^2 x}$$

We get

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$$

= tan x sec x

(iii) 5 sec x + 4 cos x

Let f (x) = 5 sec x + 4 cos x

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(5 \sec x + 4 \cos x)$$

By further calculation, we get

$$= 5\frac{d}{dx}(\sec x) + 4\frac{d}{dx}(\cos x)$$

$$= 5 \sec x \tan x + 4 \times (-\sin x)$$

$$= 5 \sec x \tan x - 4 \sin x$$

(iv) cosec x

Let f (x) = cosec x

Accordingly f (x + h) = cosec (x + h)

By first principle, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\operatorname{cosec}(x+h) - \operatorname{cosec} x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

By further calculation, we get

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right)\sin(x+h)}\right]$$

$$=-\frac{1}{\sin x}\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\times\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$=-\frac{1}{\sin x}\times 1\times\frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$=-\frac{1}{\sin x}\times\frac{\cos x}{\sin x}$$

= -cosec x cot x

(v) 3 cot x + 5 cosec x

Let f (x) = 3 cot x + 5 cosec x

f'(x) = 3 (cot x)' + 5 (cosec x)'

Let f_1 (x) = cot x,

Accordingly f_1 (x + h) = cot (x + h)

By using first principle, we get

$$f_1'(x)=\lim_{x\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right)$$

By further calculation, we get

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\ \sin(x+h)}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right)$$

$$= 1 / \sin x \ \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$

$$=-\frac{1}{\sin x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)$$

$$=-\frac{1}{\sin x}\times 1\times\frac{1}{\sin(x+0)}$$

$$=-\frac{1}{\sin^2 x}$$

$$= -\ \text{cosec}^2\ x$$

Let $f_2(x) = \text{cosec } x$,

Accordingly $f_2(x + h) = \text{cosec }(x + h)$

By using first principle, we get

$$f_2'(x)=\lim_{h\to 0}\frac{f_2(x+h)-f_2(x)}{h}$$

$$=\lim_{h\to 0}\frac{\text{cosec}(x+h)-\text{cosec}\, x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

By further calculation, we get

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right)\sin(x+h)}\right]$$

$$=-\frac{1}{\sin x}\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\times\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$=-\frac{1}{\sin x}\times 1\times\frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$=-\frac{1}{\sin x}\times\frac{\cos x}{\sin x}$$

= -cosec x cot x

Now, substitute the value of (cot x)’ and (cosec x)’ in f’(x), we get

f’(x) = 3 (cot x)’ + 5 (cosec x)’

f’(x) = 3 × (-cosec^2 x) + 5 × (-cosec x cot x)

f’(x) = -3cosec^2 x – 5cosec x cot x

(vi)5 sin x – 6 cos x + 7

Let f (x) = 5 sin x – 6 cos x + 7

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\sin(x+h)-6\cos(x+h)+7-5\sin x+6\cos x-7\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\left\{\sin(x+h)-\sin x\right\}-6\left\{\cos(x+h)-\cos x\right\}\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h)-\sin x\right]-6\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$

By further calculation, we get

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]-6\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

Now, we get

$$=5\lim_{h\to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]-6\left[(-\cos x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$

$$=5\cos x.1-6\left[(-\cos x).(0)-\sin x.1\right]$$
$$=5\cos x+6\sin x$$

(vii) 2 tan x – 7 sec x

Let f (x) = 2 tan x – 7 sec x

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\left\{\tan(x+h)-\tan x\right\}-7\left\{\sec(x+h)-\sec x\right\}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\tan(x+h)-\tan x\right]-7\lim_{h\to 0}\frac{1}{h}\left[\sec(x+h)-\sec x\right]$$

By further calculation, we get

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

Now, we get

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right)-7\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$= 2.1.\frac{1}{\cos x \cos x} - 7.1\left(\frac{\sin x}{\cos x \cos x}\right)$$
$$= 2\sec^2 x - 7\sec x \tan x$$

Miscellaneous exercise page no: 317

1. Find the derivative of the following functions from first principle:

(i) –*x*

(ii) (–*x*)–1

(iii) sin (*x* + 1)

(iv) $\cos\left(x - \frac{\pi}{8}\right)$

Solution:

(i) - x

Let f (x) = - x

Accordingly, f (x + h) = - (x + h)

Using first principle, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{-(x+h) - (-x)}{h}$$

Now, we get

$$= \lim_{h\to 0}\frac{-x - h + x}{h}$$

$$= \lim_{h\to 0}\frac{-h}{h}$$
$$= \lim_{h\to 0}(-1) = -1$$

(ii) (-x)-1

Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$

Accordingly, $f(x+h) = \frac{-1}{(x+h)}$

Using first principle, we get

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-1}{x+h} - \left(\frac{-1}{x}\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-1}{x+h} + \frac{1}{x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-x + (x+h)}{x(x+h)}\right]$$

By further calculation, we get

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-x + x + h}{x(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h}{x(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{x(x+h)}$$

$$= \frac{1}{x \cdot x}$$

= 1 / x2

(iii) sin (x + 1)

Let f (x) = sin (x + 1)

Accordingly, f (x + h) = sin (x + h + 1)

By using first principle, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\sin(x+h+1) - \sin(x+1)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h \to 0}\left[\cos\left(\frac{2x+h+2}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

We get,

$$= \lim_{h \to 0} \cos\left(\frac{2x+h+2}{2}\right) \cdot \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

We know that,

$$h \to 0 \Rightarrow \frac{h}{2} \to 0$$

$$= \cos\left(\frac{2x+0+2}{2}\right) \cdot 1$$

$$= \cos(x+1)$$

(iv) $\cos\left(x - \frac{\pi}{8}\right)$

Let $f(x)=\cos\left(x-\frac{\pi}{8}\right)$

Accordingly, $f(x+h)=\cos\left(x+h-\frac{\pi}{8}\right)$

By using first principle, we get

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]$$

We get,

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

Further we get,

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\frac{h}{2}\right]$$

So,

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right].\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$\left[\text{As } h \to 0 \Rightarrow \frac{h}{2} \to 0\right]$$

$$= -\sin\left(\frac{2x + 0 - \frac{\pi}{4}}{2}\right).1$$

Hence, we get

$$= -\sin\left(x - \frac{\pi}{8}\right)$$

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

2. (*x* + *a*)

Solution:

Let $f(x) = x + a$

Accordingly, $f(x + h) = x + h + a$

Using first principle, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

So, now we get

$$= \lim_{h \to 0} \frac{x + h + a - x - a}{h}$$

$$= \lim_{h \to 0} \left(\frac{h}{h}\right)$$

$$= \lim_{h \to 0} (1)$$

$$= 1$$

3. (px + q) (r / x + s)

Solution:

Let $f(x) = (px+q)\left(\frac{r}{x}+s\right)$

Using Leibnitz product rule, we get

$$f'(x) = (px+q)\left(\frac{r}{x}+s\right)' + \left(\frac{r}{x}+s\right)(px+q)'$$

We get,

$$= (px+q)\left(rx^{-1}+s\right)' + \left(\frac{r}{x}+s\right)(p)$$

By further calculation, we get

$$= (px+q)\left(-rx^{-2}\right) + \left(\frac{r}{x}+s\right)p$$

$$= (px+q)\left(\frac{-r}{x^2}\right) + \left(\frac{r}{x}+s\right)p$$

Now, we get

$$= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps$$

$$= ps - \frac{qr}{x^2}$$

4. (*ax* + *b*) (*cx* + *d*)2

Solution:

Let $f(x) = (ax+b)(cx+d)^2$

By using Leibnitz product rule, we get

$$f'(x) = (ax+b)\frac{d}{dx}(cx+d)^2 + (cx+d)^2\frac{d}{dx}(ax+b)$$

We get,

$$= (ax+b)\frac{d}{dx}\left(c^2x^2+2cdx+d^2\right) + (cx+d)^2\frac{d}{dx}(ax+b)$$

By differentiating separately, we get

$$= (ax+b)\left[\frac{d}{dx}(c^2x^2) + \frac{d}{dx}(2cdx) + \frac{d}{dx}d^2\right] + (cx+d)^2\left[\frac{d}{dx}ax + \frac{d}{dx}b\right]$$

So,

$$= (ax+b)(2c^2x + 2cd) + (cx + d^2)a$$
$$= 2c(ax+b)(cx+d) + a(cx+d)^2$$

5. (ax + b) / (cx + d)

Solution:

Let $f(x) = \frac{ax+b}{cx+d}$

Using quotient rule, we get

$$f'(x) = \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

Further we get

$$= \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2}$$

So, now we get

$$= \frac{acx + ad - acx - bc}{(cx+d)^2}$$

Hence,

$$= \frac{ad - bc}{(cx+d)^2}$$

6. (1 + 1 / x) / (1 – 1 / x)

Solution:

$$\text{Let } f(x) = \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}, \text{ where } x \neq 0$$

Using quotient rule, we get

$$f'(x) = \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2},\ x \neq 0,\ 1$$

Further, we get

$$= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2},\ x \neq 0,\ 1$$

So,

$$= \frac{x-1-x-1}{(x-1)^2},\ x \neq 0,\ 1$$

$$= \frac{-2}{(x-1)^2},\ x \neq 0,\ 1$$

7. 1 / (ax2 + bx + c)

Solution:

$$\text{Let } f(x) = \frac{1}{ax^2 + bx + c}$$

Using quotient rule, we get

$$f'(x) = \frac{(ax^2 + bx + c)\frac{d}{dx}(1) - \frac{d}{dx}(ax^2 + bx + c)}{(ax^2 + bx + c)^2}$$

By further calculation, we get

$$=\frac{\left(ax^2+bx+c\right)(0)-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

$$=\frac{-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

8. (ax + b) / px2 + qx + r

Solution:

$$\text{Let } f(x)=\frac{ax+b}{px^2+qx+r}$$

Using quotient rule, we get

$$f'(x)=\frac{\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)}{\left(px^2+qx+r\right)^2}$$

Further we get,

$$=\frac{\left(px^2+qx+r\right)(a)-(ax+b)(2px+q)}{\left(px^2+qx+r\right)^2}$$

Again by further calculation, we get

$$=\frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{\left(px^2+qx+r\right)^2}$$

$$=\frac{-apx^2-2bpx+ar-bq}{\left(px^2+qx+r\right)^2}$$

9. (px2 + qx + r) / ax + b

Solution:

$$\text{Let } f(x) = \frac{px^2 + qx + r}{ax + b}$$

Using quotient rule, we get

$$f'(x) = \frac{(ax+b)\frac{d}{dx}\left(px^2 + qx + r\right) - \left(px^2 + qx + r\right)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

By further calculation, we get

$$= \frac{(ax+b)(2px+q) - \left(px^2 + qx + r\right)(a)}{(ax+b)^2}$$

So, we get

$$= \frac{2apx^2 + aqx + 2bpx + bq - apx^2 - aqx - ar}{(ax+b)^2}$$

$$= \frac{apx^2 + 2bpx + bq - ar}{(ax+b)^2}$$

10. (a / x4) – (b / x2) + cox x

Solution:

$$\text{Let } f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$$

By differentiating we get,

$$f'(x) = \frac{d}{dx}\left(\frac{a}{x^4}\right) - \frac{d}{dx}\left(\frac{b}{x^2}\right) + \frac{d}{dx}(\cos x)$$

On further calculation, we get

$$= a\frac{d}{dx}\left(x^{-4}\right) - b\frac{d}{dx}\left(x^{-2}\right) + \frac{d}{dx}(\cos x)$$

We know that,

$$\left[\frac{d}{dx}\left(x^n\right) = nx^{n-1} \text{and} \frac{d}{dx}(\cos x) = -\sin x\right]$$

So,

$$= a\left(-4x^{-5}\right) - b\left(-2x^{-3}\right) + \left(-\sin x\right)$$

$$= \frac{-4a}{x^5} + \frac{2b}{x^3} - \sin x$$

11. $4\sqrt{x} - 2$

Solution:

Let $f(x) = 4\sqrt{x} - 2$

By differentiating we get,

$$f'(x) = \frac{d}{dx}\left(4\sqrt{x} - 2\right) = \frac{d}{dx}\left(4\sqrt{x}\right) - \frac{d}{dx}(2)$$

Further, we get

$$= 4\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - 0$$

$$= 4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$= \left(2x^{-\frac{1}{2}}\right)$$

$$= \frac{2}{\sqrt{x}}$$

12. (ax + b)n

Solution: na (ax + b)n-1

13. (*ax* + *b*)*n* (*cx* + *d*)*m*

Solution:

Hence, we get

$$\begin{aligned} f'(x) &= (ax+b)^n \left\{mc(cx+d)^{m-1}\right\} + (cx+d)^m \left\{na(ax+b)^{n-1}\right\} \\ &= (ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b)+na(cx+d)\right] \end{aligned}$$

14. sin (*x* + *a*)

Try yourself

15. cosec *x* cot *x*

Try youself

16. $\dfrac{\cos x}{1+\sin x}$

Solution:

Let $f(x) = \dfrac{\cos x}{1+\sin x}$

By using quotient rule, we get

$$\begin{aligned} f'(x) &= \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} \\ &= \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2} \end{aligned}$$

We get,

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

Now, we get

$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{-1}{(1+\sin x)}$$

17.

$$\frac{\sin x + \cos x}{\sin x - \cos x}$$

Solution:

Let $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(\sin x - \cos x)\frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x)\frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

On further calculation, we get

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

By expanding the terms, we get

$$= \frac{-\left[\sin^2 x + \cos^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x\right]}{(\sin x - \cos x)^2}$$

We get

$$= \frac{-[1+1]}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

18.

$$\frac{\sec x - 1}{\sec x + 1}$$

Solution:

Let $f(x) = \dfrac{\sec x - 1}{\sec x + 1}$

Now, this can be written as

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(1+\cos x)\frac{d}{dx}(1-\cos x) - (1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

On multiplying we get

$$= \frac{\sin x + \cos x \sin x + \sin x - \sin x \cos x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$

This can be written as

$$= \frac{2\sin x}{\left(1 + \frac{1}{\sec x}\right)^2}$$

On taking L.C.M we get

$$= \frac{2\sin x}{\frac{(\sec x + 1)^2}{\sec^2 x}}$$

On further calculation, we get

$$= \frac{2\sin x \sec^2 x}{(\sec x + 1)^2}$$

$$= \frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x + 1)^2}$$

$$= \frac{2\sec x \tan x}{(\sec x + 1)^2}$$

19. $\sin^n x$

Solution:

Let $y = \sin^n x$.

Accordingly, for $n = 1$, $y = \sin x$.

We know that,

$$\frac{dy}{dx} = \cos x, \text{ i.e., } \frac{d}{dx}\sin x = \cos x$$

For $n = 2$, $y = \sin^2 x$.

So, $\frac{dy}{dx} = \frac{d}{dx}(\sin x \sin x)$

By Leibnitz product rule, we get

$$= (\sin x)' \sin x + \sin x (\sin x)'$$

$$= \cos x \sin x + \sin x \cos x$$

$$= 2\sin x \cos x \qquad \ldots(1)$$

For $n = 3$, $y = \sin^3 x$.

So, $\frac{dy}{dx} = \frac{d}{dx}(\sin x \sin^2 x)$

By Leibnitz product rule, we get

$$=(\sin x)'\sin^2 x+\sin x\left(\sin^2 x\right)'$$

From equation (1) we get

$$=\cos x\sin^2 x+\sin x(2\sin x\cos x)$$

$$=\cos x\sin^2 x+2\sin^2 x\cos x$$

$$=3\sin^2 x\cos x$$

We state that, $\frac{d}{dx}\left(\sin^n x\right)=n\sin^{(n-1)}x\cos x$

For n = k, let our assertion be true

i.e., $\frac{d}{dx}\left(\sin^k x\right)=k\sin^{(k-1)}x\cos x \qquad ...(2)$

Now, consider

$$\frac{d}{dx}\left(\sin^{k+1}x\right)=\frac{d}{dx}\left(\sin x\sin^k x\right)$$

By using Leibnitz product rule, we get

$$=(\sin x)'\sin^k x+\sin x\left(\sin^k x\right)'$$

From equation (2) we get

$$=\cos x\sin^k x+\sin x\left(k\sin^{(k-1)}x\cos x\right)$$

$$=\cos x\sin^k x+k\sin^k x\cos x$$

$$=(k+1)\sin^k x\cos x$$

Hence, our assertion is true for n = k + 1

Therefore, by mathematical induction, $\frac{d}{dx}\left(\sin^n x\right)=n\sin^{(n-1)}x\cos x$

20. $\frac{a+b\sin x}{c+d\cos x}$

Solution:

Let $f(x)=\frac{a+b\sin x}{c+d\cos x}$

By differentiating and using quotient rule, we get

$$f'(x)=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$=\frac{(c+d\cos x)(b\cos x)-(a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

On multiplying we get

$$=\frac{cb\cos x+bd\cos^2 x+ad\sin x+bd\sin^2 x}{(c+d\cos x)^2}$$

Now, taking bd as common we get

$$=\frac{bc\cos x+ad\sin x+bd\left(\cos^2 x+\sin^2 x\right)}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$$

21.

$$\frac{\sin(x+a)}{\cos x}$$

Solution:

Let $f(x)=\dfrac{\sin(x+a)}{\cos x}$

By differentiating and using quotient rule, we get

$$f'(x)=\frac{\cos x\frac{d}{dx}[\sin(x+a)]-\sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x)=\frac{\cos x\frac{d}{dx}[\sin(x+a)]-\sin(x+a)(-\sin x)}{\cos^2 x} \qquad \text{... (i)}$$

Let $g(x)=\sin(x+a)$. Accordingly, $g(x+h)=\sin(x+h+a)$

By using first principle, we get

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+a)-\sin(x+a)\right]$$

On further calculation, we get

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

Now, taking limits we get

$$= \lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right).\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

We know that,

$$\left[\lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$= \left(\cos\frac{2x+2a}{2}\right)\times 1$$

$$= \cos(x+a) \qquad \text{... (ii)}$$

From equation (i) and (ii) we get

$$f'(x) = \frac{\cos x\cdot\cos(x+a)+\sin x\sin(x+a)}{\cos^2 x}$$

$$= \frac{\cos(x+a-x)}{\cos^2 x}$$

$$= \frac{\cos a}{\cos^2 x}$$

22. x^4 (5 sin x – 3 cos x)

Solution:

Let $f(x) = x^4(5\sin x - 3\cos x)$

By differentiating and using product rule, we get

$$f'(x) = x^4\frac{d}{dx}(5\sin x-3\cos x)+(5\sin x-3\cos x)\frac{d}{dx}(x^4)$$

On further calculation, we get

$$= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$

So, we get

$$= x^4\left[5\cos x - 3(-\sin x)\right] + (5\sin x - 3\cos x)(4x^3)$$

By taking x^3 as common, we get

$$= x^3\left[5x\cos x + 3x\sin x + 20\sin x - 12\cos x\right]$$

23. (x^2 + 1) cos *x*

Solution:

Let $f(x) = (x^2+1)\cos x$

By differentiating and using product rule, we get

$$f'(x) = (x^2+1)\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x^2+1)$$

On further calcualtion, we get

$$= (x^2+1)(-\sin x) + \cos x(2x)$$

By multiplying we get

$$= -x^2\sin x - \sin x + 2x\cos x$$

24. (ax^2 + sin *x*) (*p* + *q* cos *x*)

Solution:

Let $f(x) = (ax^2 + \sin x)(p + q\cos x)$

By differentiating and using product rule, we get

$$f'(x) = (ax^2+\sin x)\frac{d}{dx}(p+q\cos x) + (p+q\cos x)\frac{d}{dx}(ax^2+\sin x)$$

On further calculation, we get

$$= (ax^2+\sin x)(-q\sin x) + (p+q\cos x)(2ax+\cos x)$$

$$= -q\sin x(ax^2+\sin x) + (p+q\cos x)(2ax+\cos x)$$

25. $(x+\cos x)(x-\tan x)$

Solution:

Let $f(x)=(x+\cos x)(x-\tan x)$

By differentiating and using product rule, we get

$$f'(x)=(x+\cos x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\cos x)$$

$$=(x+\cos x)\left[\frac{d}{dx}(x)-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x)$$

Now, we get

$$=(x+\cos x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)(1-\sin x) \qquad \text{... (i)}$$

Let $g(x)=\tan x$. Accordingly, $g(x+h)=\tan(x+h)$

By using first principle, we get

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

On further calculation, we get

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

Now, we get

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]$$

So, we get

$$=\frac{1}{\cos x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right)$$

We get

$$= \frac{1}{\cos x}.1.\frac{1}{\cos(x+0)}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x \qquad \text{... (ii)}$$

Hence, from equation (i) and (ii) we get

$$f'(x) = (x+\cos x)(1-\sec^2 x) + (x-\tan x)(1-\sin x)$$

$$= (x+\cos x)(-\tan^2 x) + (x-\tan x)(1-\sin x)$$

$$= -\tan^2 x(x+\cos x) + (x-\tan x)(1-\sin x)$$

26. $\frac{4x+5\sin x}{3x+7\cos x}$

Solution:

Let $f(x) = \frac{4x+5\sin x}{3x+7\cos x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x) - (4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

On further calculation, we get

$$= \frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right] - (4x+5\sin x)\left[3\frac{d}{dx}x+7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$= \frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

On multiplying we get

$$= \frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

We get

$$= \frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$

$$= \frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

27. $\dfrac{x^2 \cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

Solution:

Let $f(x) = \dfrac{x^2 \cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

By differentiating and using quotient rule, we get

$$f'(x) = \cos\frac{\pi}{4} \cdot \left[\frac{\sin x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

By further calculation, we get

$$= \cos\frac{\pi}{4} \cdot \left[\frac{\sin x \cdot 2x - x^2 \cos x}{\sin^2 x}\right]$$

By taking x as common, we get

$$= \frac{x \cos\frac{\pi}{4}[2\sin x - x\cos x]}{\sin^2 x}$$

28. $\dfrac{x}{1+\tan x}$

Solution:

Let $f(x) = \dfrac{x}{1+\tan x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(1+\tan x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$f'(x) = \frac{(1+\tan x) - x \cdot \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \quad \text{... (i)}$$

$$f'(x) = \frac{1+\tan x - x\sec^2 x}{(1+\tan x)^2}$$

29. (x + sec x) (x – tan x)

Solution:

Let $f(x)=(x+\sec x)(x-\tan x)$

By differentiating and using product rule, we get

$$f'(x)=(x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x)$$

So, we get

$$=(x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)+\frac{d}{dx}\sec x\right]$$

$$=(x+\sec x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right] \qquad \text{... (i)}$$

Let $f_1(x)=\tan x,\ f_2(x)=\sec x$

Accordingly, $f_1(x+h)=\tan(x+h)$ and $f_2(x+h)=\sec(x+h)$

$$f_1'(x)=\lim_{h\to 0}\left(\frac{f_1(x+h)-f_1(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

By further calculation, we get

$$=\lim_{h\to 0}\left[\frac{\tan(x+h)-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

Now, by taking L.C.M we get

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$= \left(\lim_{h \to 0} \frac{\sin h}{h}\right) \cdot \left(\lim_{h \to 0} \frac{1}{\cos(x+h)\cos x}\right)$$

$$= 1 \times \frac{1}{\cos^2 x} = \sec^2 x$$

Hence we get

$$\frac{d}{dx} \tan x = \sec^2 x \qquad \text{... (ii)}$$

Now, take

$$f_2'(x) = \lim_{h \to 0} \left(\frac{f_2(x+h) - f_2(x)}{h}\right)$$

$$= \lim_{h \to 0} \left(\frac{\sec(x+h) - \sec x}{h}\right)$$

This can be written as

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right]$$

By taking L.C.M we get

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x}\right]$$

On further calculation, we get

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$

We get

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\cos(x+h)} \right]$$

By taking limits, we get

$$= \sec x . \frac{\left\{ \lim_{h \to 0} \sin\left(\frac{2x+h}{2}\right) \right\} \left\{ \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\lim_{h \to 0} \cos(x+h)}$$

We get

$$= \sec x . \frac{\sin x . 1}{\cos x}$$

$$\frac{d}{dx} \sec x = \sec x \tan x \qquad \text{.. (iii)}$$

From equation (i) (ii) and (iii) we get

$$f'(x) = (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \tan x)$$

30. $\frac{x}{\sin^n x}$

Solution:

Let $f(x) = \frac{x}{\sin^n x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

Easily, it can be shown that,

$$\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x$$

Hence,

$$f'(x)=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$$

By further calculation, we get

$$=\frac{\sin^n x.1-x\left(n\sin^{n-1} x\cos x\right)}{\sin^{2n} x}$$

By taking common terms, we get

$$=\frac{\sin^{n-1} x\left(\sin x-nx\cos x\right)}{\sin^{2n} x}$$

Hence, we get

$$=\frac{\sin x-nx\cos x}{\sin^{n+1} x}$$

UNIT 4

Imaginary Quantity

The square root of a negative real number is called an imaginary quantity or imaginary number. e.g., $\sqrt{-3}$, $\sqrt{-7/2}$
The quantity $\sqrt{-1}$ is an imaginary number, denoted by 'i', called iota.

Integral Powers of Iota (i)

$i=\sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4=1$
So, $i^{4n+1}= i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, $i^{4n+4} = i^{4n} = 1$
In other words,

$i^n = (-1)^{n/2}$, if n is an even integer
$i^n = (-1)^{(n-1)/2}.i$, if is an odd integer

Complex Number

A number of the form $z = x + iy$, where $x, y \in R$, is called a complex number

The numbers x and y are called respectively real and imaginary parts of complex number z.

i.e., $x = Re(z)$ and $y = Im(z)$

Purely Real and Purely Imaginary Complex Number

A complex number z is a purely real if its imaginary part is 0.

i.e., $Im(z) = 0$. And purely imaginary if its real part is 0 i.e., $Re(z)= 0$.

Equality of Complex Numbers

Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal, if $a_2= a_2$ and $b_1 = b_2$ i.e., $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

Algebra of Complex Numbers

1. Addition of Complex Numbers

Let $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$ be any two complex numbers, then their sum defined as
$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Properties of Addition

(i) Commutative $z_1 + z_2 = z_2 + z_1$
(ii) Associative $(z_1 + z_2) + z_3 = + (z_2 + z_3)$
(iii) Additive Identity $z + 0 = z = 0 + z$

Here, 0 is additive identity.

2. Subtraction of Complex Numbers

Let $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$ be any two complex numbers, then their difference is defined as
$z_1 – z_2 = (x_1 + iy_1) – (x_2 + iy_2)$
$= (x_1 – x_2) + i(y_1 – y_2)$

3. Multiplication of Complex Numbers

Let $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$ be any two complex numbers, then their multiplication is defined as
$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 – y_1y_2) + i(x_1y_2 + x_2y_1)$

Properties of Multiplication

(i) Commutative $z_1z_2 = z_2z_1$
(ii) Associative $(z_1 z_2) z_3 = z_1(z_2 z_3)$

(iii) Multiplicative Identity $z \cdot 1 = z = 1 \cdot z$
Here, 1 is multiplicative identity of an element z.

(iv) Multiplicative Inverse Every non-zero complex number z there exists a complex number z_1 such that $z.z_1 = 1 = z_1 \cdot z$

(v) Distributive Law
(a) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ (left distribution)
(b) $(z_2 + z_3)z_1 = z_2z_1 + z_3z_1$ (right distribution)

4. Division of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, then their division is defined as

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)}$$
$$= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

where z_2 # 0.

Conjugate of a Complex Number

If $z = x + iy$ is a complex number, then conjugate of z is denoted by z
i.e., $z = x – iy$

Properties of Conjugate

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} \Leftrightarrow z$ is purely real

(iii) $z - \bar{z} \Leftrightarrow z$ is purely imaginary

(iv) $\text{Re}(z) = \dfrac{z + \bar{z}}{2}$

(v) $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$

(vi) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(vii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(viii) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

(ix) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2},\ z_2 \neq 0$

(x) $z_1\, \bar{z}_2 \pm \bar{z}_1\, z_2 = 2\,\text{Re}(\bar{z}_1\, z_2) = 2\,\text{Re}(z_1\, \bar{z}_2)$

(xi) $(\bar{z})^n = \overline{(z^n)}$

(xii) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

(xiii) If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$

where $a_i, b_i, c_i; (i = 1, 2, 3)$ are complex numbers.

(xiv) $z\,\bar{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2$

Modulus of a Complex Number

If z = x + iy, , then modulus or magnitude of z is denoted by |z| and is given by

$|z| = x^2 + y^2$.

It represents a distance of z from origin.

In the set of complex number C, the order relation is not defined i.e., $z_1 > z_2$ or $z_i < z_2$ has no meaning but $|z_1| > |z_2|$ or $|z_1| < |z_2|$ has got its meaning, since |z| and $|z_2|$ are real numbers.

Properties of Modulus

(i) $|z| \geq 0$

(ii) If $|z| = 0$, then $z = 0$ *i.e.*, $\text{Re}(z) = 0 = \text{Im}(z)$

(iii) $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(v) $z\bar{z} = |z|^2$

(vi) $|z_1 z_2| = |z_1||z_2|$

In general

$|z_1 z_2 z_3 \ldots z_n| = |z_1||z_2||z_3| \ldots |z_n|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$

(viii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

In general

$$|z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_n| \leq |z_1| + |z_2| + |z_3| + \ldots + |z_n|$$

(ix) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

(x) $|z^n| = |z|^n$

(xi) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ greatest possible value of $|z_1 + z_2|$ is $|z_1| + |z_2|$ and least possible value of $|z_1 + z_2|$ is

$$||z_1| - |z_2||$$

(xii) $$|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$
$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$$
$$= |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2)$$
$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

(xiii) $$|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$
$$= |z_1|^2 + |z_2|^2 - (z_1\bar{z}_2 + \bar{z}_1 z_2)$$
$$= |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\bar{z}_2)$$
$$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

(xiv) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(xv) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

(xvi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

(xvii) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

where $a, b \in R$.

(xviii) z is **unimodulus**, if $|z| = 1$

Reciprocal/Multiplicative Inverse of a Complex Number

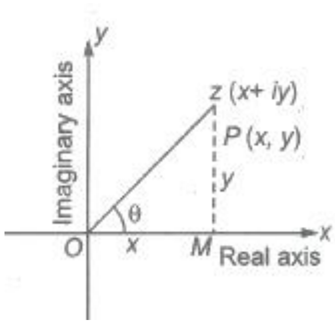
Let z = x + iy be a non-zero complex number, then

$$z^{-1} = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$
$$= \frac{x-iy}{x^2+y^2}$$
$$= \frac{x}{x^2+y^2} + \frac{i(-y)}{x^2+y^2}$$

Here, z^{-1} is called multiplicative inverse of z.

Argument of a Complex Number

Any complex number z=x+iy can be represented geometrically by a point (x, y) in a plane, called Argand plane or Gaussian plane. The angle made by the line joining point z to the origin, with the x-axis is called argument of that complex number. It is denoted by the symbol arg (z) or amp (z).



Argument (z) = θ = $\tan^{-1}$(y/x)
Argument of z is not unique, general value of the argument of z is 2nπ + θ. But arg (0) is not defined.

A purely real number is represented by a point on x-axis.

A purely imaginary number is represented by a point on y-axis.

There exists a one-one correspondence between the points of the plane and the members of the set C of all complex numbers.

The length of the line segment OP is called the modulus of z and is denoted by |z|.

i.e., length of OP = $\sqrt{x^2 + y^2}$.
Principal Value of Argument

The value of the argument which lies in the interval (- π, π] is called principal value of argument.

(i) If x> 0 and y > 0, then arg (z) = 0
(ii) If x < 0 and y> 0, then arg (z) = π -0
(iii) If x < 0 and y < 0, then arg (z) = – (π – θ)
(iv) If x> 0 and y < 0, then arg (z) = -θ

Properties of Argument

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi \ (k = 0, 1 \text{ or } -1)$

In general,

$$\arg(z_1 z_2 z_3 \dots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \dots + \arg(z_n) + 2k\pi \ (k = 0, 1 \text{ or } -1)$$

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi \ (k = 0, 1 \text{ or } = -1)$

(iv) $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$

(v) $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi \ (k = 0, 1 \text{ or } -1)$

(vi) $\arg(z^n) = n \arg(z) + 2k\pi \ (k = 0, 1 \text{ or } -1)$

(vii) If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta, k \in I$

(viii) If $\arg(z) = 0 \Rightarrow z$ is real

(ix) $\arg(z) - \arg(-z) = \begin{cases} \pi, \text{ if } \arg(z) > 0 \\ -\pi, \text{ if } \arg(z) < 0 \end{cases}$

(x) If $|z_1 + z_2| = |z_1 - z_2|$, then

$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$

(xi) If $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) = \arg(z_2)$

(xii) If $|z - 1| = |z + 1|$, then $\arg(z) = \pm\frac{\pi}{2}$

(xiii) If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, then $(z) = 1$

(xiv) If $\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{2}$, then z lies on circle of radius unity and centre at origin.

(xv) (a) If $z = 1 + \cos\theta + i\sin\theta$, then

$\arg(z) = \frac{\theta}{2}$ and $|z| = 2\cos\frac{\theta}{2}$

(b) If $z = 1 + \cos\theta - i\sin\theta$, then

$\arg(z) = -\frac{\theta}{2}$ and $|z| = 2\cos\frac{\theta}{2}$

(c) If $z = 1 - \cos\theta + i\sin\theta$, then

$\arg(z) = \frac{\pi}{2} - \frac{\theta}{2}$ and $|z| = 2\sin\frac{\theta}{2}$

(d) If $z = 1 - \cos\theta - i\sin\theta$, then

$\arg(z) = \frac{\pi}{4} - \frac{\theta}{2}$ and $|z| = \sqrt{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

(xvi) If $|z_1| \le 1, |z_2| \le 1$, then

(a) $|z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + [\arg(z_1) - \arg(z_2)]^2$

(b) $|z_1 + z_2|^2 \ge (|z_1| + |z_2|)^2 - [\arg(z_1) - \arg(z_2)]^2$

Square Root of a Complex Number

If z = x + iy, then

$$\sqrt{z} = \sqrt{x + iy} = \pm \left[\frac{\sqrt{|z| + x}}{2} + i \frac{\sqrt{|z| - x}}{2} \right], \text{for } y > 0$$

$$= \pm \left[\sqrt{\frac{|z| + x}{2}} - i \sqrt{\frac{|z| - x}{2}} \right], \text{for } y < 0$$

Polar Form

If z = x + iy is a complex number, then z can be written as

z = |z| (cos θ + i sin θ) where, θ = arg (z)

this is called polar form.

If the general value of the argument is 0, then the polar form of z is

z = |z| [cos (2nπ + θ) + i sin (2nπ + θ)], where n is an integer.

Eulerian Form of a Complex Number

If z = x + iy is a complex number, then it can be written as
$z = re^{i\theta}$, where
r = |z| and θ = arg (z)
This is called Eulerian form and $e^{i\theta}$= cosθ + i sinθ and $e^{-i\theta}$ = cosθ — i sinθ.

De-Moivre’s Theorem

A simplest formula for calculating powers of complex number known as De-Moivre’s theorem.

If n ∈ I (set of integers), then $(\cos\theta + i \sin\theta)^n$ = cos nθ + i sin nθ and if n ∈ Q (set of rational numbers), then cos nθ + i sin nθ is one of the values of $(\cos \theta + i \sin \theta)^n$.

(i) If $\frac{p}{q}$ is a rational number, then

$$(\cos\theta + i\sin\theta)^{p/q} = \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)$$

(ii) $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta - i\sin\theta)^n$

(iii) More generally, for a complex number $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\begin{aligned} z^n &= r^n(\cos\theta + i\sin\theta)^n \\ &= r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta} \end{aligned}$$

(iv) $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)\right]$

(v) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\dots(\cos\theta_n + i\sin\theta_n)$

$$= \cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n)$$

(vi) $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$

(vii) $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

Important Identities

(i) $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

(ii) $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

(iii) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(iv) $x^2 - xy + y^2 = (x + \omega y)(x + y\omega^2)$

(v) $x^2 + y^2 = (x + iy)(x - iy)$

(vi) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(vii) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

(viii) $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

or $(x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z)$

or $(x\omega + y + z\omega^2)(x\omega^2 + y + z\omega)$

(ix) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

Complex Numbers PRACTICE QUESTIONS.

Exercise 1

Q1 :

Express the given complex number in the form *a* + *ib*: $(5i)\left(-\frac{3}{5}i\right)$

Answer :

$$(5i)\left(\frac{-3}{5}i\right) = -5\times\frac{3}{5}\times i\times i$$
$$= -3i^2$$
$$= -3(-1) \qquad \left[i^2 = -1\right]$$
$$= 3$$

Q2 :

Express the given complex number in the form *a* + *ib*: $i^9 + i^{19}$

Answer :

$$i^9 + i^{19} = i^{4\times2+1} + i^{4\times4+3}$$
$$= \left(i^4\right)^2\cdot i + \left(i^4\right)^4\cdot i^3$$
$$= 1\times i + 1\times(-i) \qquad \left[i^4 = 1,\ i^3 = -i\right]$$
$$= i + (-i)$$
$$= 0$$

Q3 :

Express the given complex number in the form *a* + *ib*: i^{-39}

Answer :

$$
\begin{aligned}
i^{-39} = i^{-4\times 9-3} &= \left(i^4\right)^{-9} \cdot i^{-3} \\
&= (1)^{-9} \cdot i^{-3} \qquad \left[i^4 = 1\right] \\
&= \frac{1}{i^3} = \frac{1}{-i} \qquad \left[i^3 = -i\right] \\
&= \frac{-1}{i} \times \frac{i}{i} \\
&= \frac{-i}{i^2} = \frac{-i}{-1} = i \qquad \left[i^2 = -1\right]
\end{aligned}
$$

Q4 :

Express the given complex number in the form *a* + *ib*: 3(7 + *i*7) + *i*(7 + *i*7)

Answer :

$$
\begin{aligned}
3(7+i7)+i(7+i7) &= 21+21i+7i+7i^2 \\
&= 21+28i+7\times(-1) \qquad \left[\because i^2 = -1\right] \\
&= 14+28i
\end{aligned}
$$

Q5 :

Express the given complex number in the form *a* + *ib*: (1 - *i*) - (-1 + *i*6)

Answer :

$$
\begin{aligned}
(1-i)-(-1+i6) &= 1-i+1-6i \\
&= 2-7i
\end{aligned}
$$

Q6 :

Express the given complex number in the form *a* + *ib*: $\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$

Answer :

$$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$$
$$=\frac{1}{5}+\frac{2}{5}i-4-\frac{5}{2}i$$
$$=\left(\frac{1}{5}-4\right)+i\left(\frac{2}{5}-\frac{5}{2}\right)$$
$$=\frac{-19}{5}+i\left(\frac{-21}{10}\right)$$
$$=\frac{-19}{5}-\frac{21}{10}i$$

Q7 :

Express the given complex number in the form *a* + *ib*: $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

Answer :

$$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(\frac{-4}{3}+i\right)$$
$$=\frac{1}{3}+\frac{7}{3}i+4+\frac{1}{3}i+\frac{4}{3}-i$$
$$=\left(\frac{1}{3}+4+\frac{4}{3}\right)+i\left(\frac{7}{3}+\frac{1}{3}-1\right)$$
$$=\frac{17}{3}+i\frac{5}{3}$$

Q8 :

Express the given complex number in the form *a* + *ib*: (1 - *i*)^4

Answer :

$$
\begin{aligned}
(1-i)^4 &= \left[(1-i)^2\right]^2 \\
&= \left[1^2+i^2-2i\right]^2 \\
&= \left[1-1-2i\right]^2 \\
&= (-2i)^2 \\
&= (-2i)\times(-2i) \\
&= 4i^2 = -4 \qquad \left[i^2=-1\right]
\end{aligned}
$$

Q9 :

Express the given complex number in the form *a* + *ib*: $\left(\frac{1}{3}+3i\right)^3$

Answer :

$$
\begin{aligned}
\left(\frac{1}{3}+3i\right)^3 &= \left(\frac{1}{3}\right)^3+(3i)^3+3\left(\frac{1}{3}\right)(3i)\left(\frac{1}{3}+3i\right) \\
&= \frac{1}{27}+27i^3+3i\left(\frac{1}{3}+3i\right) \\
&= \frac{1}{27}+27(-i)+i+9i^2 \qquad \left[i^3=-i\right] \\
&= \frac{1}{27}-27i+i-9 \qquad \left[i^2=-1\right] \\
&= \left(\frac{1}{27}-9\right)+i(-27+1) \\
&= \frac{-242}{27}-26i
\end{aligned}
$$

Q10 :

Express the given complex number in the form *a* + *ib*: $\left(-2-\frac{1}{3}i\right)^3$

Answer :

$$\left(-2-\frac{1}{3}i\right)^3 = (-1)^3\left(2+\frac{1}{3}i\right)^3$$

$$= -\left[2^3 + \left(\frac{i}{3}\right)^3 + 3(2)\left(\frac{i}{3}\right)\left(2+\frac{i}{3}\right)\right]$$

$$= -\left[8 + \frac{i^3}{27} + 2i\left(2+\frac{i}{3}\right)\right]$$

$$= -\left[8 - \frac{i}{27} + 4i + \frac{2i^2}{3}\right] \qquad \left[i^3 = -i\right]$$

$$= -\left[8 - \frac{i}{27} + 4i - \frac{2}{3}\right] \qquad \left[i^2 = -1\right]$$

$$= -\left[\frac{22}{3} + \frac{107i}{27}\right]$$

$$= -\frac{22}{3} - \frac{107}{27}i$$

Q11 :

Find the multiplicative inverse of the complex number 4 - 3*i*

Answer :

Let z = 4 -3i

Then, $\bar{z}$ = 4 + 3i and $|z|^2 = 4^2 + (-3)^2 = 16 + 9 = 25$

Therefore, the multiplicative inverse of 4 -3i is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3}{25}i$$

Q12 :

Find the multiplicative inverse of the complex number $\sqrt{5} + 3i$

Answer :

Let z = $\sqrt{5} + 3i$

Then, $\bar{z} = \sqrt{5} - 3i$ and $|z|^2 = \left(\sqrt{5}\right)^2 + 3^2 = 5 + 9 = 14$

Therefore, the multiplicative inverse of $\sqrt{5}+3i$ is given by

$$z^{-1}=\frac{\bar{z}}{|z|^2}=\frac{\sqrt{5}-3i}{14}=\frac{\sqrt{5}}{14}-\frac{3i}{14}$$

Q13 :

Find the multiplicative inverse of the complex number *-i*

Answer :

Let $z = -i$

$$\text{Then, } \bar{z}=i \text{ and } |z|^2=1^2=1$$

Therefore, the multiplicative inverse of â€“*i* is given by

$$z^{-1}=\frac{\bar{z}}{|z|^2}=\frac{i}{1}=i$$

Q14 :

Express the following expression in the form of *a* + *ib*.

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

Answer :

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

$$=\frac{(3)^2-\left(i\sqrt{5}\right)^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i} \qquad \left[(a+b)(a-b)=a^2-b^2\right]$$

$$=\frac{9-5i^2}{2\sqrt{2}i}$$

$$=\frac{9-5(-1)}{2\sqrt{2}i} \qquad \left[i^2=-1\right]$$

$$=\frac{9+5}{2\sqrt{2}i}\times\frac{i}{i}$$

$$=\frac{14i}{2\sqrt{2}i^2}$$

$$=\frac{14i}{2\sqrt{2}(-1)}$$

$$=\frac{-7i}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{-7\sqrt{2}i}{2}$$

Exercise 2

Q1 :

Find the modulus and the argument of the complex number $z=-1-i\sqrt{3}$

Answer :

$z=-1-i\sqrt{3}$

Let $r\cos\theta=-1$ and $r\sin\theta=-\sqrt{3}$

On squaring and adding, we obtain

$$(r\cos\theta)^2+(r\sin\theta)^2=(-1)^2+\left(-\sqrt{3}\right)^2$$
$$\Rightarrow r^2\left(\cos^2\theta+\sin^2\theta\right)=1+3$$
$$\Rightarrow r^2=4 \qquad \left[\cos^2\theta+\sin^2\theta=1\right]$$
$$\Rightarrow r=\sqrt{4}=2 \qquad [\text{Conventionally, } r>0]$$
$$\therefore \text{Modulus}=2$$
$$\therefore\ 2\cos\theta=-1 \text{ and } 2\sin\theta=-\sqrt{3}$$
$$\Rightarrow \cos\theta=\frac{-1}{2} \text{ and } \sin\theta=\frac{-\sqrt{3}}{2}$$

Since both the values of sin $\theta$and cos θ are negative and sinθ and cosθ are negative in III quadrant,

$$\text{Argument}=-\left(\pi-\frac{\pi}{3}\right)=\frac{-2\pi}{3}$$

Thus, the modulus and argument of the complex number $-1-\sqrt{3}\,i$ are 2 and $\frac{-2\pi}{3}$ respectively.

Q2 :

Find the modulus and the argument of the complex number $z=-\sqrt{3}+i$

Answer :

$$z=-\sqrt{3}+i$$
$$\text{Let } r\cos\theta=-\sqrt{3} \text{ and } r\sin\theta=1$$

On squaring and adding, we obtain

$$r^2\cos^2\theta+r^2\sin^2\theta=\left(-\sqrt{3}\right)^2+1^2$$
$$\Rightarrow r^2=3+1=4 \qquad \left[\cos^2\theta+\sin^2\theta=1\right]$$
$$\Rightarrow r=\sqrt{4}=2 \qquad [\text{Conventionally, } r>0]$$
$$\therefore \text{Modulus}=2$$
$$\therefore\ 2\cos\theta=-\sqrt{3} \text{ and } 2\sin\theta=1$$
$$\Rightarrow \cos\theta=\frac{-\sqrt{3}}{2} \text{ and } \sin\theta=\frac{1}{2}$$
$$\therefore \theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6} \qquad [\text{As } \theta \text{ lies in the II quadrant}]$$

Thus, the modulus and argument of the complex number $-\sqrt{3}+i$ are 2 and $\frac{5\pi}{6}$ respectively.

Q3 :

Convert the given complex number in polar form: 1 - *i*

Answer :

1 - *i*

Let *r*cos *θ* = 1 and *r*sin *θ* = -1

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = 1^2 + (-1)^2$$
$$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 1+1$$
$$\Rightarrow r^2 = 2$$
$$\Rightarrow r = \sqrt{2} \qquad [\text{Conventionally, } r > 0]$$
$$\therefore \sqrt{2}\cos\theta = 1 \text{ and } \sqrt{2}\sin\theta = -1$$
$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \text{ and } \sin\theta = -\frac{1}{\sqrt{2}}$$
$$\therefore \theta = -\frac{\pi}{4} \qquad [\text{As } \theta \text{ lies in the IV quadrant}]$$
$$\therefore 1-i = r\cos\theta + i\,r\sin\theta = \sqrt{2}\cos\left(-\frac{\pi}{4}\right) + i\sqrt{2}\sin\left(-\frac{\pi}{4}\right) = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$

This is the required polar form.

Q4 :

Convert the given complex number in polar form: - 1 + *i*

Answer :

- 1 + *i*

Let *r*cos *θ* = -1 and *r*sin *θ* = 1

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = (-1)^2 + 1^2$$
$$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 1 + 1$$
$$\Rightarrow r^2 = 2$$
$$\Rightarrow r = \sqrt{2} \qquad [\text{Conventionally, } r > 0]$$
$$\therefore \sqrt{2}\cos\theta = -1 \text{ and } \sqrt{2}\sin\theta = 1$$
$$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}$$
$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \qquad [\text{As } \theta \text{ lies in the II quadrant}]$$

It can be written,

$$\therefore -1 + i = r\cos\theta + ir\sin\theta = \sqrt{2}\cos\frac{3\pi}{4} + i\sqrt{2}\sin\frac{3\pi}{4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

This is the required polar form.

Q5 :

Convert the given complex number in polar form: - 1 - *i*

Answer :

- 1 - *i*

Let $r\cos\theta = -1$ and $r\sin\theta = -1$ On

squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = (-1)^2 + (-1)^2$$
$$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 1 + 1$$
$$\Rightarrow r^2 = 2$$
$$\Rightarrow r = \sqrt{2} \qquad [\text{Conventionally, } r > 0]$$
$$\therefore \sqrt{2}\cos\theta = -1 \text{ and } \sqrt{2}\sin\theta = -1$$
$$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \sin\theta = -\frac{1}{\sqrt{2}}$$
$$\therefore \theta = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4} \qquad [\text{As } \theta \text{ lies in the III quadrant}]$$

$\therefore -1-i = r\cos\theta + ir\sin\theta = \sqrt{2}\cos\frac{-3\pi}{4} + i\sqrt{2}\sin\frac{-3\pi}{4} = \sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$ This is the required polar form.

Q6 :

Convert the given complex number in polar form: -3

Answer :

-3

Let $r\cos\theta = -3$ and $r\sin\theta = 0$

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = (-3)^2$$
$$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 9$$
$$\Rightarrow r^2 = 9$$
$$\Rightarrow r = \sqrt{9} = 3 \qquad [\text{Conventionally, } r > 0]$$
$$\therefore 3\cos\theta = -3 \text{ and } 3\sin\theta = 0$$
$$\Rightarrow \cos\theta = -1 \text{ and } \sin\theta = 0$$
$$\therefore \theta = \pi$$
$$\therefore -3 = r\cos\theta + ir\sin\theta = 3\cos\pi + i3\sin\pi = 3\left(\cos\pi + i\sin\pi\right)$$

This is the required polar form.

Q7 :

Convert the given complex number in polar form: $\sqrt{3} + i$

Answer :

$\sqrt{3} + i$

Let $r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = \left(\sqrt{3}\right)^2 + 1^2$$

$$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 3 + 1$$

$$\Rightarrow r^2 = 4$$

$$\Rightarrow r = \sqrt{4} = 2 \qquad \left[\text{Conventionally, } r > 0\right]$$

$$\therefore 2\cos\theta = \sqrt{3} \text{ and } 2\sin\theta = 1$$

$$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{6} \qquad \left[\text{As } \theta \text{ lies in the I quadrant}\right]$$

$$\therefore \sqrt{3} + i = r\cos\theta + i\,r\sin\theta = 2\cos\frac{\pi}{6} + i\,2\sin\frac{\pi}{6} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

This is the required polar form.

Q8 :

Convert the given complex number in polar form: *i*

Answer :

i

Let $r\cos\theta = 0$ and $r\sin\theta = 1$

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = 0^2 + 1^2$$

$$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 1$$

$$\Rightarrow r^2 = 1$$

$$\Rightarrow r = \sqrt{1} = 1 \qquad \left[\text{Conventionally, } r > 0\right]$$

$$\therefore \cos\theta = 0 \text{ and } \sin\theta = 1$$

$$\therefore \theta = \frac{\pi}{2}$$

$$\therefore i = r\cos\theta + i\,r\sin\theta = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

This is the required polar form.

Exercise 3

Q1 :

Solve the equation $x^2 + 3 = 0$

Answer :

The given quadratic equation is $x^2 + 3 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = 1$, $b = 0$, and $c = 3$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 0^2$, $- 4 \times 1 \times 3 = -12$ Therefore, the

required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{\pm\sqrt{-12}}{2 \times 1} = \frac{\pm\sqrt{12}\, i}{2} \qquad \left[\sqrt{-1} = i\right]$$

$$= \frac{\pm 2\sqrt{3}\, i}{2} = \pm\sqrt{3}\, i$$

Q2 :

Solve the equation $2x^2 + x + 1 = 0$

Answer :

The given quadratic equation is $2x^2 + x + 1 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = 2$, $b = 1$, and$c = 1$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 1^2$ $-4 \times 2 \times 1 = 1 - 8 = -7$ Therefore,

the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times 2} = \frac{-1 \pm \sqrt{7}\, i}{4} \qquad \left[\sqrt{-1} = i\right]$$

Q3 :

Solve the equation $x^2 + 3x + 9 = 0$

Answer :

The given quadratic equation is $x^2 + 3x + 9 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = 1$, $b = 3$, and $c = 9$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 9 = 9 - 36 = -27$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-27}}{2(1)} = \frac{-3 \pm 3\sqrt{-3}}{2} = \frac{-3 \pm 3\sqrt{3}\,i}{2} \qquad \left[\sqrt{-1} = i\right]$$

Q4 :

Solve the equation $-x^2 + x - 2 = 0$

Answer :

The given quadratic equation is $-x^2 + x - 2 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = -1$, $b = 1$, and $c = -2$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 1^2 - 4 \times (-1) \times (-2) = 1 - 8 = -7$ Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times (-1)} = \frac{-1 \pm \sqrt{7}\,i}{-2} \qquad \left[\sqrt{-1} = i\right]$$

Q5 :

Solve the equation $x^2 + 3x + 5 = 0$

Answer :

The given quadratic equation is $x^2 + 3x + 5 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = 1$, $b = 3$, and $c = 5$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 5 = 9 - 20 = -11$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-11}}{2 \times 1} = \frac{-3 \pm \sqrt{11}\, i}{2} \qquad \left[\sqrt{-1} = i\right]$$

Q6 :

Solve the equation $x^2 - x + 2 = 0$

Answer :

The given quadratic equation is $x^2 - x + 2 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = 1$, $b = -1$, and $c = 2$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = (-1)^2 - 4 \times 1 \times 2 = 1 - 8 = -7$ Therefore, the

required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-1) \pm \sqrt{-7}}{2 \times 1} = \frac{1 \pm \sqrt{7}\, i}{2} \qquad \left[\sqrt{-1} = i\right]$$

Q7 :

Solve the equation $\sqrt{2}x^2 + x + \sqrt{2} = 0$

Answer :

The given quadratic equation is $\sqrt{2}x^2 + x + \sqrt{2} = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = \sqrt{2}$, $b = 1$, and $c = \sqrt{2}$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 1^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times \sqrt{2}} = \frac{-1 \pm \sqrt{7}\, i}{2\sqrt{2}} \qquad \left[\sqrt{-1} = i\right]$$

Q8 :

Solve the equation $\sqrt{3}x^2-\sqrt{2}x+3\sqrt{3}=0$

Answer :

The given quadratic equation is $\sqrt{3}x^2-\sqrt{2}x+3\sqrt{3}=0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = \sqrt{3}$, $b = -\sqrt{2}$, and $c = 3\sqrt{3}$

Therefore, the discriminant of the given equation is

D = $b^2 - 4ac$ = $\left(-\sqrt{2}\right)^2-4\left(\sqrt{3}\right)\left(3\sqrt{3}\right)=2-36=-34$

Therefore, the required solutions are

$$\frac{-b\pm\sqrt{D}}{2a}=\frac{-\left(-\sqrt{2}\right)\pm\sqrt{-34}}{2\times\sqrt{3}}=\frac{\sqrt{2}\pm\sqrt{34}\,i}{2\sqrt{3}}\qquad\left[\sqrt{-1}=i\right]$$

Q9 :

Solve the equation $x^2+x+\frac{1}{\sqrt{2}}=0$

Answer :

The given quadratic equation is $x^2+x+\frac{1}{\sqrt{2}}=0$

This equation can also be written as $\sqrt{2}x^2+\sqrt{2}x+1=0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$a = \sqrt{2}$, $b = \sqrt{2}$, and $c = 1$

$$\therefore \text{Discriminant }(D)=b^2-4ac=\left(\sqrt{2}\right)^2-4\times\left(\sqrt{2}\right)\times1=2-4\sqrt{2}$$

Therefore, the required solutions are

$$\frac{-b\pm\sqrt{D}}{2a}=\frac{-\sqrt{2}\pm\sqrt{2-4\sqrt{2}}}{2\times\sqrt{2}}=\frac{-\sqrt{2}\pm\sqrt{2\left(1-2\sqrt{2}\right)}}{2\sqrt{2}}$$

$$=\left(\frac{-\sqrt{2}\pm\sqrt{2}\left(\sqrt{2\sqrt{2}-1}\right)i}{2\sqrt{2}}\right) \qquad \left[\sqrt{-1}=i\right]$$

$$=\frac{-1\pm\left(\sqrt{2\sqrt{2}-1}\right)i}{2}$$

Q10 :

Solve the equation $x^2+\frac{x}{\sqrt{2}}+1=0$

Answer :

The given quadratic equation is $x^2+\frac{x}{\sqrt{2}}+1=0$

This equation can also be written as $\sqrt{2}x^2+x+\sqrt{2}=0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$a = \sqrt{2}$, $b = 1$, and $c = \sqrt{2}$

$\therefore$ Discriminant $(D)=b^2-4ac=1^2-4\times\sqrt{2}\times\sqrt{2}=1-8=-7$

Therefore, the required solutions are

$$\frac{-b\pm\sqrt{D}}{2a}=\frac{-1\pm\sqrt{-7}}{2\sqrt{2}}=\frac{-1\pm\sqrt{7}\,i}{2\sqrt{2}} \qquad \left[\sqrt{-1}=i\right]$$

Exercise Miscellaneous

Q1 :

Evaluate: $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$

Answer :

$$\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$$

$$=\left[i^{4\times4+2}+\frac{1}{i^{4\times6+1}}\right]^3$$

$$=\left[\left(i^4\right)^4\cdot i^2+\frac{1}{\left(i^4\right)^6\cdot i}\right]^3$$

$$=\left[i^2+\frac{1}{i}\right]^3 \qquad \left[i^4=1\right]$$

$$=\left[-1+\frac{1}{i}\times\frac{i}{i}\right]^3 \qquad \left[i^2=-1\right]$$

$$=\left[-1+\frac{i}{i^2}\right]^3$$

$$=\left[-1-i\right]^3$$

$$=\left(-1\right)^3\left[1+i\right]^3$$

$$=-\left[1^3+i^3+3\cdot1\cdot i\left(1+i\right)\right]$$

$$=-\left[1+i^3+3i+3i^2\right]$$

$$=-\left[1-i+3i-3\right]$$

$$=-\left[-2+2i\right]$$

$$=2-2i$$

Q2 :

For any two complex numbers z_1 and z_2, prove that

Re (z_1z_2) = Re z_1 Re z_2 - Im z_1 Im z_2

Answer :

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$\therefore z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$$
$$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2 y_1y_2$$
$$= x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2 \qquad [i^2 = -1]$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$
$$\Rightarrow \operatorname{Re}(z_1z_2) = x_1x_2 - y_1y_2$$
$$\Rightarrow \operatorname{Re}(z_1z_2) = \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2$$

Hence, proved.

Q3 :

Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ **to the standard form.**

Answer :

$$\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \left[\frac{(1+i)-2(1-4i)}{(1-4i)(1+i)}\right]\left[\frac{3-4i}{5+i}\right]$$
$$= \left[\frac{1+i-2+8i}{1+i-4i-4i^2}\right]\left[\frac{3-4i}{5+i}\right] = \left[\frac{-1+9i}{5-3i}\right]\left[\frac{3-4i}{5+i}\right]$$
$$= \left[\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}\right] = \frac{33+31i}{28-10i} = \frac{33+31i}{2(14-5i)}$$
$$= \frac{(33+31i)}{2(14-5i)} \times \frac{(14+5i)}{(14+5i)} \qquad \left[\text{On multiplying numerator and denominator by } (14 + 5i)\right]$$
$$= \frac{462+165i+434i+155i^2}{2\left[(14)^2-(5i)^2\right]} = \frac{307+599i}{2(196-25i^2)}$$
$$= \frac{307+599i}{2(221)} = \frac{307+599i}{442} = \frac{307}{442} + \frac{599i}{442}$$

This is the required standard form.

Q4 :

If x â€“ iy = $\sqrt{\frac{a-ib}{c-id}}$ prove that $\left(x^2+y^2\right)^2=\frac{a^2+b^2}{c^2+d^2}$.

Answer :

$$x-iy=\sqrt{\frac{a-ib}{c-id}}$$

$$=\sqrt{\frac{a-ib}{c-id}\times\frac{c+id}{c+id}}\left[\text{On multiplying numerator and deno min ator by }(c+id)\right]$$

$$=\sqrt{\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}}$$

$$\therefore(x-iy)^2=\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

$$\Rightarrow x^2-y^2-2ixy=\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

On comparing real and imaginary parts, we obtain

$$x^2-y^2=\frac{ac+bd}{c^2+d^2},\quad -2xy=\frac{ad-bc}{c^2+d^2}\qquad(1)$$

$$\left(x^2+y^2\right)^2=\left(x^2-y^2\right)^2+4x^2y^2$$

$$=\left(\frac{ac+bd}{c^2+d^2}\right)^2+\left(\frac{ad-bc}{c^2+d^2}\right)^2\qquad\left[\text{Using }(1)\right]$$

$$=\frac{a^2c^2+b^2d^2+2acbd+a^2d^2+b^2c^2-2adbc}{\left(c^2+d^2\right)^2}$$

$$=\frac{a^2c^2+b^2d^2+a^2d^2+b^2c^2}{\left(c^2+d^2\right)^2}$$

$$=\frac{a^2\left(c^2+d^2\right)+b^2\left(c^2+d^2\right)}{\left(c^2+d^2\right)^2}$$

$$=\frac{\left(c^2+d^2\right)\left(a^2+b^2\right)}{\left(c^2+d^2\right)^2}$$

$$=\frac{a^2+b^2}{c^2+d^2}$$

Hence, proved.

Q5 :

Convert the following in the polar form:

(i) $\frac{1+7i}{(2-i)^2}$ **, (ii)** $\frac{1+3i}{1-2i}$

Answer :

(i) Here, $z=\frac{1+7i}{(2-i)^2}$

$$=\frac{1+7i}{(2-i)^2}=\frac{1+7i}{4+i^2-4i}=\frac{1+7i}{4-1-4i}$$

$$=\frac{1+7i}{3-4i}\times\frac{3+4i}{3+4i}=\frac{3+4i+21i+28i^2}{3^2+4^2}$$

$$=\frac{3+4i+21i-28}{3^2+4^2}=\frac{-25+25i}{25}$$

$$=-1+i$$

Let $r\cos\theta=-1$ and $r\sin\theta=1$ On

squaring and adding, we obtain

$r^2(\cos^2\theta+\sin^2\theta)=1+1$

$\Rightarrow r^2(\cos^2\theta+\sin^2\theta)=2$

$\Rightarrow r^2=2$ $\quad$ $[\cos^2\theta+\sin^2\theta=1]$

$$\Rightarrow r=\sqrt{2} \qquad [\text{Conventionally, } r>0]$$

$$\therefore \sqrt{2}\cos\theta=-1 \text{ and } \sqrt{2}\sin\theta=1$$

$$\Rightarrow \cos\theta=\frac{-1}{\sqrt{2}} \text{ and } \sin\theta=\frac{1}{\sqrt{2}}$$

$$\therefore \theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4} \qquad [\text{As } \theta \text{ lies in II quadrant}]$$

$\therefore z=r\cos\theta+i\,r\sin\theta$

$$=\sqrt{2}\cos\frac{3\pi}{4}+i\sqrt{2}\sin\frac{3\pi}{4}=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

This is the required polar form.

(ii) Here, $z=\frac{1+3i}{1-2i}$

$$= \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i}$$
$$= \frac{1+2i+3i-6}{1+4}$$
$$= \frac{-5+5i}{5} = -1+i$$

Let $r \cos \theta = -1$ and $r \sin \theta = 1$ On

squaring and adding, we obtain

$r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$
$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 2$

$\Rightarrow r^2 = 2$ $\quad$ $[\cos^2 \theta + \sin^2 \theta = 1]$

$$\Rightarrow r = \sqrt{2} \qquad [\text{Conventionally, } r > 0]$$
$$\therefore \sqrt{2} \cos \theta = -1 \text{ and } \sqrt{2} \sin \theta = 1$$
$$\Rightarrow \cos \theta = \frac{-1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}}$$
$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \qquad [\text{As } \theta \text{ lies in II quadrant}]$$

$\therefore z = r \cos \theta + i\, r \sin \theta$

$$= \sqrt{2} \cos \frac{3\pi}{4} + i \sqrt{2} \sin \frac{3\pi}{4} = \sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$$

This is the required polar form.

Q6 :

Solve the equation $3x^2 - 4x + \frac{20}{3} = 0$

Answer :

The given quadratic equation is $3x^2 - 4x + \frac{20}{3} = 0$

This equation can also be written as $9x^2 - 12x + 20 = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$a = 9$, $b = -12$, and $c = 20$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = (-12)^2 - 4 \times 9 \times 20 = 144 - 720 = -576$

Q7 :

Solve the equation $x^2 - 2x + \frac{3}{2} = 0$

Answer :

The given quadratic equation is $x^2 - 2x + \frac{3}{2} = 0$

This equation can also be written as $2x^2 - 4x + 3 = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$a = 2$, $b =$ â€“4, and $c = 3$

Therefore, the discriminant of the given equation is

D = $b^2 - 4ac = (-4)^2 - 4 \times 2 \times 3 = 16 - 24 = -8$ Therefore, the

required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-4) \pm \sqrt{-8}}{2 \times 2} = \frac{4 \pm 2\sqrt{2}\,i}{4} \qquad \left[\sqrt{-1} = i\right]$$

$$= \frac{2 \pm \sqrt{2}\,i}{2} = 1 \pm \frac{\sqrt{2}}{2} i$$

Q8 :

Solve the equation $27x^2 - 10x + 1 = 0$

Answer :

The given quadratic equation is $27x^2 - 10x + 1 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = 27$, $b = -10$, and $c = 1$

Therefore, the discriminant of the given equation is

D = $b^2 - 4ac = (-10)^2 - 4 \times 27 \times 1 = 100 - 108 = -8$

Q9 :

Solve the equation $21x^2 - 28x + 10 = 0$

Answer :

The given quadratic equation is $21x^2 - 28x + 10 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = 21$, $b = -28$, and $c = 10$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = (-28)^2 - 4 \times 21 \times 10 = 784 - 840 = -56$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-28) \pm \sqrt{-56}}{2 \times 21} = \frac{28 \pm \sqrt{56}\, i}{42}$$

$$= \frac{28 \pm 2\sqrt{14}\, i}{42} = \frac{28}{42} \pm \frac{2\sqrt{14}}{42} i = \frac{2}{3} \pm \frac{\sqrt{14}}{21} i$$

Q10 :

If $z_1 = 2 - i,\ z_2 = 1 + i,$ **find** $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$**.**

Answer :

$$z_1 = 2 - i,\ z_2 = 1 + i$$

$$\therefore \left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right| = \left|\frac{(2-i)+(1+i)+1}{(2-i)-(1+i)+1}\right|$$

$$= \left|\frac{4}{2-2i}\right| = \left|\frac{4}{2(1-i)}\right|$$

$$= \left|\frac{2}{1-i} \times \frac{1+i}{1+i}\right| = \left|\frac{2(1+i)}{1^2 - i^2}\right|$$

$$= \left|\frac{2(1+i)}{1+1}\right| \qquad \left[i^2 = -1\right]$$

$$= \left|\frac{2(1+i)}{2}\right|$$

$$= |1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Thus, the value of $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$ is $\sqrt{2}$.

Q11 :

If $z_1 = 2 - i,\ z_2 = 1 + i,$ **find** $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$.

Answer :

$$z_1 = 2-i,\ z_2 = 1+i$$

$$\therefore \left|\frac{z_1+z_2+1}{z_1-z_2+1}\right| = \left|\frac{(2-i)+(1+i)+1}{(2-i)-(1+i)+1}\right|$$

$$= \left|\frac{4}{2-2i}\right| = \left|\frac{4}{2(1-i)}\right|$$

$$= \left|\frac{2}{1-i}\times\frac{1+i}{1+i}\right| = \left|\frac{2(1+i)}{1^2-i^2}\right|$$

$$= \left|\frac{2(1+i)}{1+1}\right| \qquad \left[i^2=-1\right]$$

$$= \left|\frac{2(1+i)}{2}\right|$$

$$= |1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

Thus, the value of $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$ is $\sqrt{2}$.

Q12 :

If *a* + *ib* = $\frac{(x+i)^2}{2x^2+1}$ **, prove that** a^2+b^2 **=** $\frac{(x^2+1)^2}{(2x+1)^2}$

Answer :

$$a+ib = \frac{(x+i)^2}{2x^2+1}$$

$$= \frac{x^2+i^2+2xi}{2x^2+1}$$

$$= \frac{x^2-1+i2x}{2x^2+1}$$

$$= \frac{x^2-1}{2x^2+1} + i\left(\frac{2x}{2x^2+1}\right)$$

On comparing real and imaginary parts, we obtain

$$a=\frac{x^2-1}{2x^2+1}\text{ and }b=\frac{2x}{2x^2+1}$$

$$\therefore a^2+b^2=\left(\frac{x^2-1}{2x^2+1}\right)^2+\left(\frac{2x}{2x^2+1}\right)^2$$

$$=\frac{x^4+1-2x^2+4x^2}{(2x+1)^2}$$

$$=\frac{x^4+1+2x^2}{(2x^2+1)^2}$$

$$=\frac{(x^2+1)^2}{(2x^2+1)^2}$$

$$\therefore a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}$$

Hence, proved.

Q13 :

Let $z_1=2-i,\ z_2=-2+i$**. Find**

(i) $\mathrm{Re}\left(\frac{z_1z_2}{\overline{z}_1}\right)$**, (ii)** $\mathrm{Im}\left(\frac{1}{z_1\overline{z}_1}\right)$

Answer :

$z_1=2-i,\ z_2=-2+i$

(i) $z_1z_2=(2-i)(-2+i)=-4+2i+2i-i^2=-4+4i-(-1)=-3+4i$

$$\overline{z}_1=2+i$$

$$\therefore\frac{z_1z_2}{\overline{z}_1}=\frac{-3+4i}{2+i}$$

On multiplying numerator and denominator by (2 â€“ *i*), we obtain

$$\frac{z_1z_2}{\overline{z}_1}=\frac{(-3+4i)(2-i)}{(2+i)(2-i)}=\frac{-6+3i+8i-4i^2}{2^2+1^2}=\frac{-6+11i-4(-1)}{2^2+1^2}$$

$$=\frac{-2+11i}{5}=\frac{-2}{5}+\frac{11}{5}i$$

On comparing real parts, we obtain

$$\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = \frac{-2}{5}$$

(ii) $\dfrac{1}{z_1\bar{z}_1} = \dfrac{1}{(2-i)(2+i)} = \dfrac{1}{(2)^2+(1)^2} = \dfrac{1}{5}$

On comparing imaginary parts, we obtain

$$\text{Im}\left(\frac{1}{z_1\bar{z}_1}\right) = 0$$

Q14 :

Find the modulus and argument of the complex number $\dfrac{1+2i}{1-3i}$ **.**

Answer :

Let $z = \dfrac{1+2i}{1-3i}$, then

$$z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1^2+3^2} = \frac{1+5i+6(-1)}{1+9}$$

$$= \frac{-5+5i}{10} = \frac{-5}{10} + \frac{5i}{10} = \frac{-1}{2} + \frac{1}{2}i$$

Let $z = r\cos\theta + ir\sin\theta$

i.e., $r\cos\theta = \dfrac{-1}{2}$ and $r\sin\theta = \dfrac{1}{2}$

On squaring and adding, we obtain

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$\Rightarrow r^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow r = \frac{1}{\sqrt{2}} \qquad \left[\text{Conventionally, } r > 0\right]$$

$$\therefore \frac{1}{\sqrt{2}}\cos\theta = \frac{-1}{2} \text{ and } \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2}$$

$$\Rightarrow \cos\theta = \frac{-1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \qquad \left[\text{As } \theta \text{ lies in the II quadrant}\right]$$

Therefore, the modulus and argument of the given complex number are $\frac{1}{\sqrt{2}}$ and $\frac{3\pi}{4}$ respectively.

Q15 :

Find the real numbers *x* and *y* if (*x* - *iy*) (3 + 5*i*) is the conjugate of -6 - 24*i*.

Answer :

Let $z = (x - iy)(3 + 5i)$

$$z = 3x + 5xi - 3yi - 5yi^2 = 3x + 5xi - 3yi + 5y = (3x + 5y) + i(5x - 3y)$$

$$\therefore \bar{z} = (3x + 5y) - i(5x - 3y)$$

It is given that, $\bar{z} = -6 - 24i$

$$\therefore (3x + 5y) - i(5x - 3y) = -6 - 24i$$

Equating real and imaginary parts, we obtain

$$3x + 5y = -6 \qquad \text{... (i)}$$

$$5x - 3y = 24 \qquad \text{... (ii)}$$

Multiplying equation (i) by 3 and equation (ii) by 5 and then adding them, we obtain

$$\begin{array}{r} 9x + 15y = -18 \\ \underline{25x - 15y = 120} \\ 34x \quad = 102 \end{array}$$

$$\therefore x = \frac{102}{34} = 3$$

Putting the value of *x* in equation (i), we obtain

$$3(3) + 5y = -6$$

$$\Rightarrow 5y = -6 - 9 = -15$$

$$\Rightarrow y = -3$$

Thus, the values of *x* and *y* are 3 and â€“3 respectively.

Q16 :

Find the modulus of $\frac{1+i}{1-i}-\frac{1-i}{1+i}$**.**

Answer :

$$\frac{1+i}{1-i}-\frac{1-i}{1+i}=\frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}$$

$$=\frac{1+i^2+2i-1-i^2+2i}{1^2+1^2}$$

$$=\frac{4i}{2}=2i$$

$$\therefore\left|\frac{1+i}{1-i}-\frac{1-i}{1+i}\right|=|2i|=\sqrt{2^2}=2$$

Q17 :

If (*x* + *iy*)^3 **= *u* + *iv*, then show that** $\frac{u}{x}+\frac{v}{y}=4\left(x^2-y^2\right)$**.**

Answer :

$$(x+iy)^3=u+iv$$

$$\Rightarrow x^3+(iy)^3+3\cdot x\cdot iy(x+iy)=u+iv$$

$$\Rightarrow x^3+i^3y^3+3x^2yi+3xy^2i^2=u+iv$$

$$\Rightarrow x^3-iy^3+3x^2yi-3xy^2=u+iv$$

$$\Rightarrow\left(x^3-3xy^2\right)+i\left(3x^2y-y^3\right)=u+iv$$

On equating real and imaginary parts, we obtain

$$u = x^3 - 3xy^2,\ v = 3x^2y - y^3$$

$$\therefore \frac{u}{x} + \frac{v}{y} = \frac{x^3 - 3xy^2}{x} + \frac{3x^2y - y^3}{y}$$

$$= \frac{x\left(x^2 - 3y^2\right)}{x} + \frac{y\left(3x^2 - y^2\right)}{y}$$

$$= x^2 - 3y^2 + 3x^2 - y^2$$

$$= 4x^2 - 4y^2$$

$$= 4\left(x^2 - y^2\right)$$

$$\therefore \frac{u}{x} + \frac{v}{y} = 4\left(x^2 - y^2\right)$$

Hence, proved.

Q18 :

If α and ÃŽÂ² are different complex numbers with $|\beta|$ = 1, then find $\left|\frac{\beta - \alpha}{1 - \overline{\alpha}\beta}\right|$.

Answer :

Let α = *a* + *ib* and ÃŽÂ² = *x* + *iy*

It is given that, $|\beta| = 1$

$$\therefore \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow x^2 + y^2 = 1 \qquad \text{... (i)}$$

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=\left|\frac{(x+iy)-(a+ib)}{1-(a-ib)(x+iy)}\right|$$

$$=\left|\frac{(x-a)+i(y-b)}{1-(ax+aiy-ibx+by)}\right|$$

$$=\left|\frac{(x-a)+i(y-b)}{(1-ax-by)+i(bx-ay)}\right|$$

$$=\frac{|(x-a)+i(y-b)|}{|(1-ax-by)+i(bx-ay)|} \qquad \left[\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\right]$$

$$=\frac{\sqrt{(x-a)^2+(y-b)^2}}{\sqrt{(1-ax-by)^2+(bx-ay)^2}}$$

$$=\frac{\sqrt{x^2+a^2-2ax+y^2+b^2-2by}}{\sqrt{1+a^2x^2+b^2y^2-2ax+2abxy-2by+b^2x^2+a^2y^2-2abxy}}$$

$$=\frac{\sqrt{(x^2+y^2)+a^2+b^2-2ax-2by}}{\sqrt{1+a^2(x^2+y^2)+b^2(y^2+x^2)-2ax-2by}}$$

$$=\frac{\sqrt{1+a^2+b^2-2ax-2by}}{\sqrt{1+a^2+b^2-2ax-2by}} \qquad [\text{Using } (1)]$$

$$=1$$

$$\therefore\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=1$$

Q19 :

Find the number of non-zero integral solutions of the equation $|1-i|^x=2^x$ **.**

Answer :

$$|1-i|^x = 2^x$$

$$\Rightarrow \left(\sqrt{1^2+(-1)^2}\right)^x = 2^x$$

$$\Rightarrow \left(\sqrt{2}\right)^x = 2^x$$

$$\Rightarrow 2^{\frac{x}{2}} = 2^x$$

$$\Rightarrow \frac{x}{2} = x$$

$$\Rightarrow x = 2x$$

$$\Rightarrow 2x - x = 0$$

$$\Rightarrow x = 0$$

Thus, 0 is the only integral solution of the given equation. Therefore, the number of non-zero integral solutions of the given equation is 0.

Q20 :

If (a + ib) (c + id) (e + if) (g + ih) = A + iB, then show that

($a^2 + b^2$) ($c^2 + d^2$) ($e^2 + f^2$) ($g^2 + h^2$) = $A^2 + B^2$.

Answer :

$$(a+ib)(c+id)(e+if)(g+ih) = A + iB$$

$$\therefore |(a+ib)(c+id)(e+if)(g+ih)| = |A+iB|$$

$$\Rightarrow |(a+ib)|\times|(c+id)|\times|(e+if)|\times|(g+ih)| = |A+iB| \qquad [|z_1z_2| = |z_1||z_2|]$$

$$\Rightarrow \sqrt{a^2+b^2}\times\sqrt{c^2+d^2}\times\sqrt{e^2+f^2}\times\sqrt{g^2+h^2} = \sqrt{A^2+B^2}$$

On squaring both sides, we obtain

($a^2 + b^2$) ($c^2 + d^2$) ($e^2 + f^2$) ($g^2 + h^2$) = $A^2 + B^2$

Hence, proved.

Q21 :

If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m.

Answer :

$$\left(\frac{1+i}{1-i}\right)^m = 1$$

$$\Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = 1$$

$$\Rightarrow \left(\frac{(1+i)^2}{1^2+1^2}\right)^m = 1$$

$$\Rightarrow \left(\frac{1^2+i^2+2i}{2}\right)^m = 1$$

$$\Rightarrow \left(\frac{1-1+2i}{2}\right)^m = 1$$

$$\Rightarrow \left(\frac{2i}{2}\right)^m = 1$$

$$\Rightarrow i^m = 1$$

$\therefore m = 4k,$ where k is some integer.

Therefore, the least positive integer is 1.

Thus, the least positive integral value of *m* is 4 (= 4 × 1).

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